SUR LA NOTION DE DIFFÉRENTIELLE D'UNE FONCTION DE LIGNE*

PAR

MAURICE FRÉCHET

TABLE DES MATIÈRES

		Page
§ 1.	Introduction	135
2–4.	Aperçu historique	136
5.	Cas des fonctions ordinaires de plusieurs variables	138
6.	Différentielle d'une fonctionnelle	139
7.	Diverses formes des fonctionnelles linéaires	139
8, 9.	Différentielles des fonctions de lignes continues	141
10.	Expression de la variation	142
11, 12.	La différentielle d'une fonction de ligne n'est pas une fonctionnelle linéaire arbitraire des accroissements des coordonnées	143
13.	Forme explicite de la différentielle	146
14.	Forme géométrique de la différentielle	149
15, 16.	Remarque	150
17–21.	Parties régulières et irrégulières de la différentielle	152
22.	Autre forme de la différentielle	155
23–25.	Transformation de la différentielle en intégrale de M. Lebesgue	157

Introduction

1. Avant d'étudier le cas particulier des fonctions de lignes, il m'a paru utile de montrer que la définition de la différentielle que je propose s'inspire de considérations générales applicables à une fonctionnelle quelconque. Je commence donc aux §§ 2, 3 par rappeler les points de vue de M. Volterra et de M. Hadamard. Ces auteurs se sont occupés uniquement de la dérivée des fonctionnelles telle qu'on l'entend dans le Calcul des Variations. Je pense qu'il y a lieu de définir d'abord la *différentielle* en généralisant la définition donnée par Stolz dans le cas d'une fonction ordinaire de plusieurs variables. J'arrive ainsi aux §§ 4, 5, 6, 7 à une définition générale de la différentielle des fonctionnelles qu'il suffit de préciser dans chaque cas particulier.

Je m'arrête ensuite au cas des fonctions de lignes et je déduis (§§ 8, 9) l'expression de la " variation " de celle de la différentielle. Cette différentielle est une fonctionnelle linéaire des accroissements des coordonnées des points

* Presented to the Society September 9, 1913.

de la courbe. Mais cette fonctionnelle linéaire n'est pas arbitraire et satisfait à une condition que j'obtiens aux §§ 10, 11.

Je mets ensuite la fonctionnelle linéaire sous la forme de somme d'intégrales de Stieltjes,—forme qui se déduit immédiatement d'un théorème de F. Riesz. On peut ainsi écrire explicitement (§§ 11, 12) la condition à laquelle j'ai fait allusion. Elle permet de mettre (§§ 13, 14, 15, 16) l'expression de la différentielle sous une forme qui n'est plus assujettie à aucune condition d'égalité —dans le cas où la courbe est rectifiable et à tangente continue—

$$\int_a^b (y_0' \Delta z - z_0' \Delta y)\, dp + \int_a^b (z_0' \Delta x - x_0' \Delta z)\, dq + \int_a^b (x_0' \Delta y - y_0' \Delta x)\, dr$$

$$+ A\Delta x(a) + B\Delta y(a) + C\Delta z(a) + A'\Delta x(b) + B'\Delta y(b) + C'\Delta z(b),$$

où p, q, r sont des fonctions à variation bornée quelconques, où A, B, C, A', B', C' sont des constantes quelconques et où les intégrales sont prises au sens de Stieltjes.

J'applique ensuite le mode de décomposition—que j'ai présenté ailleurs—d'une fonctionnelle linéaire, pour mettre en évidence (§§ 17, 18, 19, 20, 21, 22) des termes réguliers et irréguliers dans la différentielle d'une fonction de ligne.

Enfin, j'indique comment en suivant une méthode indiquée par M. Lebesgue on peut, en compliquant l'élément d'intégration, se passer des intégrales de Stieltjes (§§ 23–25). Toute cette théorie est établie en supposant que les seuls voisinages imposés soient d'ordre zéro au sens adopté en Calcul des Variations. Les résultats seraient un peu plus compliqués si l'on imposait aux voisinages d'être du premier ordre.

Aperçu historique

2. Le premier essai pour appliquer aux fonctionnelles les procédés du Calcul Différentiel semble être dû à M. Volterra. Le point de départ de sa méthode est la définition de la dérivée d'une fonctionnelle. Cette définition est l'extension de celle qui s'était imposée dans le Calcul des Variations. Dans le cas où l'argument de la fonctionnelle U_L est une ligne continue L plane par exemple, M. Volterra suppose que si l'on déforme un peu L au voisinage d'un point M de L, la variation de U_L ainsi produite est un infiniment petit d'ordre au moins égal à celui de l'aire balayée dans la déformation et que la limite du quotient ne dépend que de L et de M. Cette supposition se trouve être exacte dans les cas classiques du Calcul des Variations, au moins entre les extrémités de L; et elle conduit à la même formule pour la variation de U_L, soit une formule telle que

$$\delta U_L = \int_L U_{L,x}\, \delta y dx \tag{1}$$

où $U_{L,x}$ dépend de la ligne L et du point de L d'abscisse x, mais non de la variation δy de l'ordonnée en ce point.

3. M. Volterra ne manqua pas de remarquer qu'une telle définition n'était pas entièrement satisfaisante, puisqu'elle laisse de côté une grande partie des expressions qui interviennent dans le Calcul des Variations, à savoir celles des variations des intégrales définies où les limites ne sont pas fixes. De telles variations comportent en effet, outre une intégrale définie de la forme (1), des termes finis aux limites. Il convint donc d'ajouter au second membre de (1) des termes qui dépendent, *d'une manière spéciale*, selon son expression, de certains points exceptionnels. En adoptant la définition de M. Volterra, on s'inspire des premierès applications qui se sont présentées et que M. Volterra a traitées avec un succés qui justifie pratiquement sa définition. Mais il était souhaitable au point de vue logique et pour assurer le développement futur de la théorie de déduire la définition d'un principe unique et général. M. Hadamard proposa donc de " considérer comme fonctionnelles auxquelles on peut étendre les méthodes du Calcul Infinitésimal, toutes les fonctionnelles U_y dont la variation est une fonctionnelle linéaire de la variation de y." (La définition d'une fonctionnelle linéaire sera donnée plus tard (§ 6)).

4. Je me propose de poursuivre encore plus loin la critique de M. Hadamard. Notons d'abord en passant qu'une fonctionnelle linéaire peut comporter encore un terme d'une nature différente des deux termes mentionnés plus haut (intégrale et terme fini); nous retrouverons cette remarque au § 7. Mais surtout observons que la première chose à faire pour appliquer au Calcul Fonctionnel les méthodes du Calcul Infinitésimal, c'est de définir la *différentielle* et d'en tirer seulement comme conséquence l'expression de la *variation*. On comprendra notre pensée en l'appliquant au cas des fonctions ordinaires de plusieurs variables numériques. C'est en effet seulement après avoir défini la différentielle totale de telles fonctions qu'il y a lieu de donner la règle de dérivation des fonctions composées.

Nous serons donc conduit à essayer de définir d'abord la notion de différentielle, et pour cela à nous poser deux questions: Quelle doit être la forme de la différentielle? De quelle manière doit elle se déduire de la fonctionnelle donnée? La réponse à la première question nous est fournie par la remarque de M. Hadamard que le résultat fondamental du Calcul Différentiel est le suivant: " La différentielle d'une fonction est une fonction linéaire des différentielles des variables." Nous supposerons donc que la différentielle d'une fonctionnelle est une fonctionnelle linéaire—nous préciserons plus loin (§ 6) le sens de cette expression—de l'accroissement de l'argument. La remarque de M. Hadamard suffirait si nous désirions seulement arriver à une définition de la " variation." Car la réponse à la seconde question est contenue dans la définition de la variation: c'est la dérivée par rapport à α de la fonction de α qui est la valeur

de la fonctionnelle quand l'argument dépend aussi de α. Mais la réponse est moins immédiate quand nous voulons arriver à définir une différentielle totale. Ici les dérivées partielles n'ont plus d'analogues. J'ai donc été amené à essayer de reprendre l'ancienne définition aujourd'hui généralement abandonnée: la différentielle est la partie principale de l'accroissement de la fonction quand l'accroissement de la variable est considéré comme infiniment petit.

Cas des fonctions ordinaires de plusieurs variables

5. Il faut bien remarquer en effet que si cette ancienne définition—qui est la première en date, la plus naturelle et la plus commode—a prêté à des objections très fondées, c'est qu'elle a été mal formulée. Tout au contraire, c'est à elle que sont revenus des auteurs désireux de plus de rigueur dans la théorie des fonctions de plusieurs variables. MM. Stolz,* Pierpont,† W. H. Young,‡ formulent ainsi la définition de la différentielle d'une fonction de plusieurs variables:

Une fonction $f(x, y)$ admet une différentielle totale au point x_0, y_0 si (1) $f(x, y)$ admet en ce point des dérivées partielles en x et en y, *et si* (2) *on a, quels que soient les accroissements* Δx, Δy,

$$(2)\quad f(x_0+\Delta x, y_0+\Delta y) = f(x_0, y_0) + \Delta x \frac{\partial f}{\partial x_0} + \Delta y \frac{\partial f}{\partial y_0} + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

ϵ_1 et ϵ_2 tendant vers zéro quand Δx, Δy tendent simultanément vers zéro. Cette différentielle est alors

$$\Delta x \frac{\partial f}{\partial x_0} + \Delta y \frac{\partial f}{\partial y_0}.$$

Sans connaître cette nouvelle définition, j'ai été amené en appliquant les idées exposées plus haut à la définition suivante qui lui est entièrement équivalente mais dont la forme se rapproche plus de la définition historique:

Une fonction $f(x, y)$ admet au point x_0, y_0 une différentielle totale s'il existe une fonction linéaire des accroissements des variables, soit $A\Delta x + B\Delta y$, qui ne diffère de l'accroissement de la fonction que par un infiniment petit relativement à la distance $\sqrt{\Delta x^2 + \Delta y^2}$ du point x_0, y_0 au point voisin $x_0 + \Delta x$, $y_0 + \Delta y$. Au lieu de $\sqrt{\Delta x^2 + \Delta y^2}$, il est équivalent de prendre $|\Delta x| + |\Delta y|$, de sorte que d'après ma définition

$$(3)\quad \frac{f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) - [A\Delta x + B\Delta y]}{|\Delta x| + |\Delta y|} = \epsilon,$$

ϵ tendant vers zéro avec $|\Delta x| + |\Delta y|$. Il est facile de voir que dans ce

* *Grundzüge der Differential- und Integralrechnung*, t. 1, p. 133.

† *The Theory of Functions of Real Variables*, t. 1, p. 268.

‡ *The Fundamental Theorems of Differential Calculus.*

cas, on a $A = \partial f / \partial x_0$, $B = \partial f / \partial y_0$ et que la formule (3) est équivalente à la formule (2) de Stolz.

M. W. H. Young a, le premier, nettement montré les simplifications que procurent la définition nouvelle.* J'ai moi-même repris et complété cette exposition.† De sorte que je pourrai admettre ici comme établie la supériorité de la définition de Stolz sur la définition ordinaire (où l'on omet généralement la condition (2) de Stolz).

Différentielle d'une fonctionelle

6. Il nous faut maintenant revenir à la définition de la différentielle d'une fonctionnelle. J'ai d'abord considéré dans les Comptes Rendus du Congrès des Sociétés Savantes (Paris, 1912, p. 44) le cas des fonctionnelles dont l'argument est une fonction continue. Je rappelle brièvement les définitions et théorèmes concernant ce cas et qui nous serviront dans la suite.

Une fonctionnelle linéaire est une fonctionnelle distributive et continue. Dans le cas actuel:

Une fonctionnelle U_f définie dans le champ C des fonctions $f(x)$ continues dans (a, b) est continue dans ce champ si U_{f_n} tend vers U_f lorsque $f_n(x)$ converge *uniformément* vers $f(x)$.

Elle est distributive si l'on a identiquement $U_{f_1+f_2} = U_{f_1} + U_{f_2}$. Nous dirons qu'une fonctionnelle U_f définie dans la champ C a une différentielle pour l'argument f_0 s'il existe une fonctionnelle linéaire $T_{\Delta f}$ de l'accroissement Δf de l'argument, qui ne diffère de l'accroissement de la fonctionnelle U_f que par une quantité infiniment petite par rapport au maximum de la valeur absolue de l'accroissement $\Delta f(x)$ dans (a, b). En appelant $m\Delta f$ ce maximum, on aura donc

$$U_{f+\Delta f} = U_f + T_{\Delta f} + \epsilon m\Delta f, \tag{4}$$

ϵ tendant vers zéro avec $m\Delta f$.

Diverses formes des fonctionnelles linéaires

7. Quant à la forme des fonctionnelles linéaires, on peut facilement prévoir l'existence de deux types différents parmi ces fonctionnelles, l'un tel que

$$\int_a^b f(x)\,\alpha(x)\,dx \tag{5}$$

où $\alpha(x)$ est une fonction sommable‡ quelconque, indépendante de l'argument

* Loc. cit.

† Nouvelles Annales de Mathématiques, 4e série, t. 12 (1912), pp. 385, 433.

‡ Dans tout le cours de cet article, les intégrales ordinaires seront prises au sens de M. Lebesgue.

$f(x)$; l'autre pouvant s'écrire

$$\sum_n A_n f(c_n) \tag{6}$$

où les A et c sont des constantes indépendantes de l'argument $f(x)$ de la fonctionnelle, où les c sont entre a et b, et la série ΣA_n est absolument convergente. Mais il existe des fonctionnelles linéaires qui n'appartiennent à aucun de ces deux types.

La représentation générale d'une fonctionnelle linéaire *quelconque* U_f peut s'effectuer sous les trois formes équivalentes suivantes. D'après M. Hadamard, on peut écrire

$$U_f = \lim_{n=\infty} \int_a^b f(x)\, \alpha_n(x)\, dx, \tag{7}$$

où les $\alpha_n(x)$ sont des fonctions indépendantes de $f(x)$, qui peuvent être continues et qu'on peut même astreindre à être des polynomes. D'après M. Lebesgue, on a

$$U_f = \int_a^b f[\varphi(x)]\, \beta(x)\, dx, \tag{8}$$

où $\varphi(x)$ et $\beta(x)$ sont des fonctions indépendantes de l'argument f, $\varphi(x)$ étant une fonction non décroissante et $\beta(x)$ une fonction sommable qui prend seulement les valeurs ± 1. Enfin la représentation qui paraît la plus commode, puisqu'elle détermine la fonctionnelle au moyen d'une seule fonction ordinaire $u(x)$, est celle de M. F. Riesz,

$$U_f = \int_a^b f(x)\, d[u(x)], \tag{9}$$

où $u(x)$ est une fonction à variation bornée et où l'intégrale est prise au sens de Stieltjes. C'est à dire que l'on a

$$U_f = \lim \sum_i f(\xi_i)\, [u(x_i) - u(x_{i-1})], \tag{10}$$

où l'on a pris arbitrairement les x et ξ tels que $a = x_0 \leqq \xi_1 \leqq x_1 \leqq \xi_2 \leqq \cdots \leqq \xi_n \leqq x_n = b$ et où l'on fait tendre vers zéro la longueur maximum des intervalles (x_{i-1}, x_i).

Il faut bien remarquer cependant que si l'on veut se rendre compte de la structure d'une fonctionnelle linéaire, il n'est pas nécessaire d'employer l'intégrale de Stieltjes sous sa forme la plus générale, c'est à dire en prenant pour $u(x)$ une fonction quelconque à variation bornée. J'ai montré en effet* que toute fonctionnelle linéaire peut se mettre *et d'une seule* manière sous la forme d'une somme de trois fonctionnelles linéaires de types essentiellement

* M. Fréchet, *Sur la Notion de Différentielle dans le Calcul Fonctionnel*, Comptes Rendus du Congrès des Sociétés Savantes (Grenoble, 1913).

différents et représentables sous les formes respectives (5), (6) et (9), où la fonction $u(x)$ qui figure dans (9) doit non seulement être à variation bornée mais encore posséder une dérivée nulle presque partout (c'est à dire sauf en un ensemble de points de mesure nulle). Un exemple de ce troisième type nous est fourni par la fonctionnelle

$$U_f = \lim_{\text{mesure } S_E = 0} \left[\frac{\int_{S_E} f(x)\, dx}{\int_{S_E} dx} \right]$$

où S_E est un ensemble dénombrable d'intervalles non empiétant et couvrant l'ensemble parfait de mesure nulle E.

Différentielles des fonctions de lignes continues

8. J'appellerai *ligne continue* toute suite ordonnée de points équivalente au point de vue de l'Analysis Situs à un segment de droite et par conséquent représentable sous la forme

$$(L) \qquad x = f(t), \quad y = g(t), \quad z = h(t), \quad a \leqq t \leqq b,$$

où f, g, h sont trois fonctions arbitraires uniformément continues dans (a, b) et qui ne sont à la fois constantes dans aucun intervalle de valeurs de t.

Si à toute ligne continue L correspond un nombre bien déterminé U_L, on dit que U_L est une fonction de ligne.

9. On voit que la donnée d'une ligne équivaut à celle d'un ensemble de trois fonctions continues de sorte que les définitions de la continuité et de la différentiation d'une fonction de ligne se déduisent immédiatement des définitions correspondantes pour les fonctionnelles dont l'argument est une seule fonction continue dans (a, b). Il est bon cependant de les énoncer explicitement pour éviter tout malentendu.

Une fonctionnelle $U_{f, g, h}$ dépendant de trois fonctions $f(t)$, $g(t)$, $h(t)$ uniformément continues dans (a, b) est continue en (f_0, g_0, h_0) si $U_{f, g, h}$ tend vers U_{f_0, g_0, h_0} de quelque manière que f, g, h convergent uniformément et *simultanément* vers f_0, g_0, h_0. Elle est distributive si l'on a identiquement

$$U_{f+f_1, g+g_1, h+h_1} \equiv U_{f, g, h} + U_{f_1, g_1, h_1}.$$

Enfin la fonctionnelle $U_{f, g, h}$ admet une différentielle en (f_0, g_0, h_0) s'il existe une fonctionnelle $T_{\Delta f, \Delta g, \Delta h}$ linéaire par rapport à l'ensemble des accroissements $\Delta f, \Delta g, \Delta h$ des arguments, qui ne diffère de l'accroissement de la fonctionnelle que par une quantité infiniment petite par rapport à l'ensemble des accroissements $\Delta f, \Delta g, \Delta h$. Pour préciser, nous supposons

$$(11) \qquad \frac{U_{f_0+\Delta f, g_0+\Delta g, h_0+\Delta h} - U_{f_0, g_0, h_0} - T_{\Delta f, \Delta g, \Delta h}}{m\Delta f + m\Delta g + m\Delta h} = \epsilon,$$

$m\Delta f$, $m\Delta g$, $m\Delta h$ désignant les maxima dans (a, b) des valeurs absolues $|\Delta f|$, $|\Delta g|$, $|\Delta h|$, et ϵ tendant vers zéro quand la somme de ces maxima tend vers zéro. On peut aussi remplacer le dénominateur par $m\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$; la définition n'en est pas altérée.

Nous pouvons maintenant remarquer que, si $V_{f,g,h}$ est une fonctionnelle linéaire par rapport à l'ensemble (f, g, h), c'est la somme de trois fonctionnelles linéaires dépendant chacune d'une seule des fonctions f, g, h. Car on a

$$V_{f,g,h} = V_{f,0,0} + V_{0,g,0} + V_{0,0,h}.$$

Si maintenant on décompose ainsi la différentielle $T_{\Delta f,\Delta g,\Delta h}$ d'une fonctionnelle $U_{f,g,h}$ en trois fonctionnelles linéaires

$$T_{\Delta f,\Delta g,\Delta h} = \mathscr{T}^{(f)}_{\Delta f} + \mathscr{T}^{(g)}_{\Delta g} + \mathscr{T}^{(h)}_{\Delta h}, \tag{12}$$

la formule (11) montre immédiatement que la fonctionnelle $U_{f,g,h}$ admet respectivement ces trois fonctionnelles comme différentielles par rapport à f seul, g seul, ou h seul. Mais la réciproque n'est pas vraie.

Expression de la variation

10. Nous pouvons passer maintenant au calcul de la "variation" de la fonctionnelle $U_{f,g,h}$. Supposons que f, g, h soient fonctions non seulement de la variable t, mais d'un paramètre α. Alors U sera lui-même une fonction de α

$$F(\alpha) = U_{f(t,\alpha),\, g(t,\alpha),\, h(t,\alpha)}.$$

Si pour $\alpha = \alpha_0$, $f(t, \alpha)$, $g(t, \alpha)$, $h(t, \alpha)$ se réduisent à $f_0(t)$, $g_0(t)$, $h_0(t)$, la variation de U pour f_0, g_0, h_0 sera la différentielle de F par rapport à α pour $\alpha = \alpha_0$. Pour pouvoir calculer cette variation, il va falloir faire certaines hypothèses sur $f(t, \alpha)$, $g(t, \alpha)$, $h(t, \alpha)$. En effet dans le cas classique et plus simple où $g \equiv h \equiv 0$ et où

$$U_{f(t,\alpha)} = \int_a^b f(t, \alpha)\, dt,$$

on a déjà à introduire ces hypothèses. On suppose généralement que $f(t, \alpha)$ est dérivable par rapport à α pour $\alpha = \alpha_0$ et que $[f(t, \alpha) - f(t, \alpha_0)]/(\alpha - \alpha_0)$ converge uniformément vers $\partial f/\partial\alpha_0$ quand α tend vers α_0. *Nous ferons donc cette hypothèse pour* $f(t, \alpha)$ *ainsi que pour* $g(t, \alpha)$ *et* $h(t, \alpha)$. *Elle nous suffira en supposant bien entendu que* $U_{f,g,h}$ *admet une différentielle pour* f_0, g_0, h_0. On aura en effet d'après (11) et (12)

$$\frac{F(\alpha)-F(\alpha_0)}{\alpha-\alpha_0}=\mathscr{T}^{(f_0)}_{\frac{f(t,\alpha)-f(t,\alpha_0)}{\alpha-\alpha_0}}+\mathscr{T}^{(g_0)}_{\frac{g(t,\alpha)-g(t,\alpha_0)}{\alpha-\alpha_0}}+\mathscr{T}^{(h_0)}_{\frac{h(t,\alpha)-h(t,\alpha_0)}{\alpha-\alpha_0}}$$
$$+\epsilon\left\{m\left[\frac{f(t,\alpha)-f(t,\alpha_0)}{\alpha-\alpha_0}\right]+m\left[\frac{g(t,\alpha)-g(t,\alpha_0)}{\alpha-\alpha_0}\right]\right.$$
$$\left.+m\left[\frac{h(t,\alpha)-h(t,\alpha_0)}{\alpha-\alpha_0}\right]\right\}.$$

D'après les hypothèses faites $\partial f/\partial\alpha_0$, $\partial g/\partial\alpha_0$, $\partial h/\partial\alpha_0$ sont des fonctions continues, et le second membre de l'équation précédente tendra vers

$$\mathscr{T}^{(f_0)}_{\frac{\delta f}{\delta\alpha_0}}+\mathscr{T}^{(g_0)}_{\frac{\delta g}{\delta\alpha_0}}+\mathscr{T}^{(h_0)}_{\frac{\delta h}{\delta\alpha_0}}.$$

Donc $F(\alpha)$ a une dérivée en α_0 et cette dérivée est $T_{\delta f/\delta\alpha_0,\ \delta g/\delta\alpha_0,\ \delta h/\delta\alpha_0}$.

Avec les notations ordinaires du Calcul des Variations:

$$\delta U_{f_0,\,g_0,\,h_0}=T_{\delta f_0,\,\delta g_0,\,\delta h_0}$$

ou encore

$$\delta U_L=T_{\delta x,\,\delta y,\,\delta z}. \tag{13}$$

Autrement dit, *on obtient, sous les conditions indiquées plus haut, la "variation" d'une fonction de ligne en remplaçant dans l'expression de sa différentielle les accroissements des coordonnées par leurs variations.*

La différentielle d'une fonction de ligne n'est pas une fonctionnelle linéaire arbitraire des accroissements des coordonnées

11. Si tout système de fonctions f, g, h continues et non à la fois constantes définit bien une courbe L, la réciproque n'est pas vraie; tous les systèmes obtenus en faisant dans f, g, h une substitution de la forme $t=\theta(t')$, où $\theta(t')$ est une fonction continue de t' qui croît de la valeur a à la valeur b quand t croît de a à b définit la même courbe L. On peut donc s'attendre à ce que, pour une ligne L_0 déterminée, la différentielle de U ne soit pas une fonctionnelle linéaire quelconque des Δf, Δg, Δh. C'est ce que nous allons montrer. Bornons-nous pour simplifier au cas très général où la courbe L_0 est rectifiable et a partout une tangente variant continuement, c'est à dire, où $f_0(t)$, $g_0(t)$, $h_0(t)$ ont des dérivées uniformément continues dans (a, b) et telles que $f_0'^2+g_0'^2+h_0'^2$ reste différent de zéro; et appliquons la formule (13) lorsqu'on prend pour ligne variée la ligne L_0 elle-même. Considérons dans ce but une fonction $\varphi(t)$ continue entre a et b, nulle en a et b, et admettant une dérivée continue dans (a, b). Pour λ assez petit, la fonction $\theta(t)=t+\lambda\varphi(t)$ est continue et croît de a à b quand t varie de a à b. Dès lors la courbe

$$x=f_0[\theta(t)],\qquad y=g_0[\theta(t)],\qquad z=h_0[\theta(t)]$$

représente la même courbe L_0, et les seconds membres convergent uniformément vers $f_0(t)$, $g_0(t)$, $h_0(t)$ quand λ tend vers zéro. Mais de plus $\{f_0[\theta(t)] - f_0(t)\}/\lambda$, $\{g_0[\theta(t)] - g_0(t)\}/\lambda$, $\{h_0[\theta(t)] - h_0(t)\}/\lambda$ convergeront uniformément vers $\varphi(t) f_0'(t)$, $\varphi(t) g_0'(t)$, $\varphi(t) h_0'(t)$; nous aurons donc le droit d'appliquer la formule (13) où δU sera évidemment nul et nous aurons donc

$$T_{\phi f_0', \phi g_0', \phi h_0'} = 0. \tag{14}$$

Ainsi le fait que la fonctionnelle U est une fonction de ligne se traduit par l'égalité (14) qui a lieu quelle que soit la fonction $\varphi(t)$ nulle aux extrémités et à dérivée continue.

12. Pour mettre cette condition (14) sous une forme indépendante de la fonction arbitraire $\varphi(t)$, nous utiliserons la représentation (9) des fonctionnelles linéaires due à M. F. Riesz. La fonctionnelle T est la somme de trois fonctionnelles linéaires et pourra donc s'écrire

$$T_{\Delta f, \Delta g, \Delta h} = \int_a^b \Delta f(t)\, d[u(t)] + \int_a^b \Delta g(t)\, d[v(t)] + \int_a^b \Delta h(t) d[w(t)], \tag{15}$$

u, v, w désignant trois fonctions à variation bornées qui ne dépendent que de f_0, g_0, h_0, U. On aura donc

$$0 = \int_a^b \varphi(t) f_0'(t) d[u(t)] + \int_a^b \varphi(t) g_0'(t) d[v(t)] + \int_a^b \varphi(t) h_0'(t) d[w(t)]. \tag{14'}$$

Posons

$$F(t) = \int_a^t f_0'(t)\, d[u(t)].$$

Pour étudier ses discontinuités, introduisons la variation totale $U(t)$ de $u(t)$ dans (a, b) et le maximum $\Omega(\epsilon)$ de l'oscillation de la fonction continue $f_0'(t)$ dans tout intervalle de longueur $\leqq \epsilon$. On aura évidemment

$$\begin{aligned} \int_{t_{i-1}}^{t_i} f_0'(t)\, d[u(t)] &= \lim \Sigma f_0'(\xi_j')\,[u(t_j') - u(t_{j-1}')] \\ &= \lim \{\Sigma [f_0'(\xi_j') - f_0'(\xi)]\,[u(t_j') - u(t_{j-1}')] \\ &\qquad + f_0'(\xi)\,\Sigma [u(t_j') - u(t_{j-1}')]\} \end{aligned} \tag{16}$$

si $t_{i-1} = t_0' \leqq \xi_1' \leqq t_1' \leqq \cdots \leqq t_n' = t_i$ et $t_{i-1} \leqq \xi \leqq t_i$. D'où

$$\left| \int_{t_{i-1}}^{t_i} f_0'(t)\, d[u(t)] - f_0'(\xi)\,[u(t_i) - u(t_{i-1})] \right| \leqq \Omega(\epsilon)\,|U(t_i) - U(t_{i-1})|, \tag{17}$$

ϵ étant supérieur ou égal à la longueur de l'intervalle $(t_i - t_{i-1})$. Si maintenant on fait tendre t_{i-1} vers $t' = t_i$ par valeurs plus petites ou t_i vers $t' = t_{i-1}$ par valeurs plus grandes, on voit qu'on aura

$$F(t' \pm 0) - F(t') = f_0'(t')\,[u(t' \pm 0) - u(t)]. \tag{18}$$

D'autre part, l'égalité (16) donne évidemment

$$\Sigma \mid F(t_i) - F(t_{i-1}) \mid \leqq M \cdot U(b),$$

M étant le maximum de $f_0'(t)$ dans (a, b). Donc $F(t)$ est à variation bornée dans (a, b). Par suite, on peut écrire

$$\int_a^b \varphi(t) f_0'(t)\, d[u(t)] = \int_a^b \varphi(t)\, d[F(t)], \tag{19}$$

car le second membre a un sens et le premier est, d'après (17), la limite de

$$\begin{aligned}\Sigma\varphi(\xi_i) f_0'(\xi_i)[u(t_i) - u(t_{i-1})] &= \Sigma\varphi(\xi_i)[F(t_i) - F(t_{i-1})] \\ &\quad + \theta q \Omega(\epsilon_0)\, U(b),\end{aligned} \tag{20}$$

avec $\mid \theta \mid < 1$, ϵ_0 représentant la longueur du plus grand des intervalles $(t_i - t_{i-1})$, et q le maximum de $\varphi(t)$. Or le second membre de (20) tend vers le second membre de (19) quand ϵ_0 tend vers zero.

Ceci étant, posons

$$K(t) = \int_a^t x_0'(t)\, d[u(t)] + \int_a^t y_0'(t)\, d[v(t)] + \int_a^t z_0'(t)\, d[w(t)] \tag{21}$$

en ecrivant x_0, y_0, z_0 pour f_0, g_0, h_0. On verra comme pour $F(t)$ que c'est une fonction à variation bornée et que l'égalité (14′) peut s'écrire

$$\int_a^b \varphi(t)\, d[K(t)] = 0.$$

Or on voit facilement que

$$\int_a^b \varphi(t)\, d[K(t)] = [\varphi(t) K(t)]_a^b - \int_a \varphi'(t) K(t)\, dt,$$

et puisque $\varphi(b) = \varphi(a) = 0$, l'égalité (14′) est finalement équivalente à

$$\int_a^b \varphi'(t) K(t)\, dt = 0.$$

Cette condition, vérifiée quelle que soit la fonction $\varphi(t)$ nulle aux extrémités et à dérivée continue, ne peut être satisfaite si $K(t)$ est arbitraire. Remarquons d'abord qu'au moyen d'un changement de variable simple, on peut supposer $a = 0$, $b = 2\pi$ et qu'alors on aura le droit de prendre pour $\varphi(\theta)$, θ étant la nouvelle variable, l'une des fonctions

$$\sin n\theta, \qquad \cos n\theta - 1$$

et par suite pour $\varphi'(t)$ l'une des fonctions

$$n \cos n\theta, \qquad -n \sin n\theta,$$

n étant un entier quelconque. Si donc $n \neq 0$, on aura

$$\int_0^{2\pi} K_1(\theta) \cos n\theta \, d\theta = \int_0^{2\pi} K_1(\theta) \sin n\theta \, d\theta = 0 \quad (n = 1, 2, 3 \cdots),$$

où $K_1(\theta) = K(t)$. La série de Fourier de $K_1(\theta)$ se réduit donc à une constante, et d'autre part, on sait que la série de Fourier d'une fonction à variation bornée $K_1(\theta)$ converge partout vers $K_1(\theta)$ sauf peut-être aux points de discontinuité de $K_1(\theta)$ qui forment un ensemble dénombrable.

En revenant à la variable primitive t, on voit donc que $K(t)$ est égal en tout point de (a, b) à une constante déterminée k, sauf peut-être aux points d'un certain ensemble dénombrable N. Il est d'ailleurs facile de déterminer N. En raisonnant pour $K(t)$ comme pour $F(t)$, on obtient en effet l'égalité analogue à (18),

$$K(t \pm 0) - K(t) = x_0'(t)[u(t \pm 0) - u(t)] + \cdots + z_0'(t)[w(t \pm 0) - w(t)]. \tag{22}$$

Par suite $K(t)$ ne peut être discontinue qu'aux points où l'une des fonctions u, v, w est discontinue.

Ainsi lorsque la fonction de ligne U_L admet une différentielle pour une ligne $L_0[x_0(t), y_0(t), z_0(t)]$ à tangente continue, si l'on met cette différentielle sous la forme

$$\int_a^b \Delta x(t) \, d[u(t)] + \cdots + \int_a^b \Delta z(t) \, d[w(t)], \tag{23}$$

les fonctions à variation bornée u, v, w ne peuvent être arbitraires; elles vérifient nécessairement l'égalité

$$\int^t x_0'(t) \, d[u(t)] + \cdots + \int^t z_0'(t) \, d[w(t)] = \text{constante} \tag{24}$$

en tout point t de (a, b)—sauf peut-être en un ensemble dénombrable de points qui ne peuvent être que des points de discontinuité de u, v ou w.

Forme explicite de la différentielle

13. La forme (23) de la différentielle est très utile en ce sens qu'elle détermine la différentielle seulement au moyen de trois fonctions u, v, w, et que réciproquement ces trois fonctions sont déterminées par la différentielle—du moins, d'après F. Riesz, à un changement près de leur valeur en un ensemble dénombrable de points autres que a et b. Mais elle a l'inconvénient que ces trois fonctions u, v, w soient assujetties à la condition (24). Nous allons donner une autre expression de la différentielle où les fonctions qui déterminent la différentielle sont des fonctions à variation bornée choisies arbitrairement.

Nous avons supposé que x_0', y_0', z_0' sont des fonctions continues de t et que $x_0'^2 + y_0'^2 + z_0'^2$ a un minimum positif h. Par suite les expressions

$$p(t) = \int_a^t \frac{y_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[w(t)] - \int_a^t \frac{z_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[v(t)];$$

$$(25)\quad q(t) = \int_a^t \frac{z_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[u(t)] - \int_a^t \frac{x_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[w(t)];$$

$$r(t) = \int_a^t \frac{x_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[v(t)] - \int_a^t \frac{y_0'(t)}{x_0'^2 + y_0'^2 + z_0'^2} d[u(t)]$$

sont calculables comme intégrales de Stieltjes. De plus, en raisonnant comme au § 12 pour la fonction $F(t)$, on voit que p, q, r sont des fonctions à variation bornée dans (a, b). Ceci étant, calculons la différence

$$\begin{aligned}\delta = \int_a^b \Delta x(t)\, d[u(t)] + \int_a^b \Delta y(t)\, d[v(t)] + \int_a^b \Delta z(t)\, d[w(t)] \\ - \int_a^b [y_0'(t)\Delta z(t) - z_0'(t)\Delta y(t)]\, d[p(t)] \\ - \int_a^b [z_0'(t)\Delta x(t) - x_0'(t)\Delta z(t)]\, d[q(t)] \\ - \int_a^b [x_0'(t)\Delta y(t) - y_0'(t)\Delta x(t)]\, d[r(t)].\end{aligned}$$

C'est une somme algébrique d'intégrales de Stieltjes et c'est par suite la limite de trois termes tels que

$$(26)\quad \sum_i \Delta x(\xi_i)\{[u(t_i) - u(t_{i-1})] + y_0'(\xi_i)[r(t_i) - r(t_{i-1})] - z_0'(\xi_i)[q(t_i) - q(t_{i-1})]\}.$$

L'accolade peut s'écrire

$$\begin{aligned}u(t_i) - u(t_{i-1}) - \int_{t_{i-1}}^{t_i} \frac{x_0'(t)x_0'(\xi_i) + y_0'(t)y_0'(\xi_i) + z_0'(t)z_0'(\xi_i)}{x_0'^2(t) + y_0'^2(t) + z_0'^2(t)} d[u(t)] \\ (27)\quad + \int_{t_{i-1}}^{t_i} \frac{x_0'(t)}{x_0'^2(t) + y_0'^2(t) + z_0'^2(t)} \{x_0'(\xi_i)\, d[u(t)] + y_0'(\xi_i)\, d[v(t)] + z_0'(\xi_i)\, d[w(t)]\}.\end{aligned}$$

D'autre part, remarquons que si l'on prend pour t_{i-1} et t_i des points où u, v, w sont continues, on aura d'après (24)

$$\int_{t_{i-1}}^{t_i} x_0'(t)\, d[u(t)] + \int_{t_{i-1}}^{t_i} y_0'(t)\, d[v(t)] + \int_{t_{i-1}}^{t_i} z_0'(t)\, d[w(t)] = 0.$$

Retranchons de (27) cette expression multipliée par

$$x_0'(\xi_i)/[x_0'^2(\xi_i)+y_0'^2(\xi_i)+z_0'^2(\xi_i)].$$

L'expression (27) peut donc s'écrire

$$\int_{t_{i-1}}^{t_i}\left\{\frac{x_0'(t)[x_0'(t)-x_0'(\xi_i)]+y_0'(t)[y_0'(t)-y_0'(\xi_i)]+z_0'(t)[z_0'(t)-z_0'(\xi_i)]}{x_0'^2(t)+y_0'^2(t)+z_0'^2(t)}\right\}d[u(t)]$$

$$+\int_{t_{i-1}}^{t_i}\left[\frac{x_0'(t)\,x_0'(\xi_i)}{x_0'^2(t)+y_0'^2(t)+z_0'^2(t)}-\frac{x_0'(t)\,x_0'(\xi_i)}{x_0'^2(\xi_i)+y_0'^2(\xi_i)+z_0'^2(\xi_i)}\right]du$$

(28)

$$+\int_{t_{i-1}}^{t_i}\left[\frac{x_0'(t)\,y_0'(\xi_i)}{x_0'^2(t)+y_0'^2(t)+z_0'^2(t)}-\frac{y_0'(t)\,x_0'(\xi_i)}{x_0'^2(\xi_i)+y_0'^2(\xi_i)+z_0'^2(\xi_i)}\right]dv$$

$$+\int_{t_{i-1}}^{t_i}\left[\frac{x_0'(t)\,z_0'(\xi_i)}{x_0'^2(t)+y_0'^2(t)+z_0'^2(t)}-\frac{z_0'(t)\,x_0'(\xi_i)}{x_0'^2(\xi_i)+y_0'^2(\xi_i)+z_0'^2(\xi_i)}\right]dw.$$

Soit maintenant M le plus grand maximum de $|x_0'(t)|$, $|y_0'(t)|$, $|z_0'(t)|$ dans (a,b); $\omega(\epsilon)$ la plus grande oscillation de $x_0'(t)$, $y_0'(t)$, $z_0'(t)$ dans un intervalle de longueur au plus égale à la plus grande longueur ϵ des intervalles (t_{i-1},t_i); $U(t)$, $V(t)$, $W(t)$ les variations totales de u, v, w dans (a,t). D'après la définition des intégrales de Stieltjes, on a en général

$$\left|\int_{t_{i-1}}^{t_i}x(t)\,d[u(t)]\right|\leqq[\text{maximum de }|x(t)|\text{ dans }(t_{i-1},t_i)]\times[U(t_i)-U(t_{i-1})].$$

En appliquant une formule analogue à chacune des intégrales que nous venons d'écrire, on voit que la valeur absolue de l'expression (27) est au plus égale à

$$\frac{3M\omega(\epsilon)}{h}[U(t_i)-U(t_{i-1})]+\frac{6M^3\omega(\epsilon)}{h^2}[U(t_i)-U(t_{i-1})]$$
$$+\frac{11M^3\omega(\epsilon)}{h^2}[V(t_i)-V(t_{i-1})]+\frac{11M^3\omega(\epsilon)}{h^2}[W(t_i)-W(t_{i-1})].$$

Ceci ne s'applique que si t_i et t_{i-1} ne sont pas de points de discontinuité de u, v ou w. Si l'on suppose d'abord que a et b ne sont pas de tels points de discontinuité, on pourra s'arranger pour que tous les t_i soient ainsi choisis et la valeur absolue de la somme des trois termes analogues à (26) sera au plus égale à

$$N\left[\frac{3M\omega(\epsilon)}{h}+\frac{6M^3\omega(\epsilon)}{h^2}+\frac{22M^3\omega(\epsilon)}{h^2}\right][U(b)+V(b)+W(b)]=\omega(\epsilon)\times K$$

où N désigne le plus grand des maxima de $|\Delta x(t)|$, $|\Delta y(t)|$, $|\Delta z(t)|$

dans (a, b), et K un nombre fixe. Quand ϵ tend vers zéro, $\omega(\epsilon) \times K$ tend vers zéro. Donc δ serait nulle. Mais il peut arriver que u, v ou w soit discontinu en a ou b. Alors nous aurons seulement prouvé que l'on a

$$\delta = \lim_{\epsilon=0} \sum_i \left[\frac{\Delta x(\xi_i)\, x_0'(\xi_i) + \Delta y(\xi_i)\, y_0'(\xi_i) + \Delta z(\xi_i)\, z_0'(\xi_i)}{x_0'^2(\xi_i) + y_0'^2(\xi_i) + z_0'^2(\xi_i)} \right] \times [K(t_i) - K(t_{i-1})].$$

En prenant comme précédemment les t_i distincts de a et de b, il restera seulement les termes extrêmes et on aura à la limite

$$\delta = \frac{\Delta x(b)\, x_0'(b) + \Delta y(b)\, y_0'(b) + \Delta z(b)\, z_0'(b)}{x_0'^2(b) + y_0'^2(b) + z_0'^2(b)} [K(b) - K(b-0)]$$
$$+ \frac{\Delta x(a)\, x_0'(a) + \Delta y(a)\, y_0'(a) + \Delta z(a)\, z_0'(a)}{x_0'^2(a) + y_0'^2(a) + z_0'^2(a)} [K(a+0) - K(a)].$$

En définitive *nous obtenons maintenant l'expression suivante de la différentielle d'une fonction de ligne:*

$$\int_a^b [y_0' \Delta z - z_0' \Delta y]\, dp + \int_a^b [z_0' \Delta x - x_0' \Delta z]\, dq + \int_a^b [x_0' \Delta y - y_0' \Delta x]\, dr$$
$$+ R[x_0'(a) \Delta x(a) + y_0'(a) \Delta y(a) + z_0'(a) \Delta z(a)]$$
$$+ S[x_0'(b) \Delta x(b) + y_0'(b) \Delta y(b) + z_0'(b) \Delta z(b)], \tag{29}$$

où $p(t)$, $q(t)$, $r(t)$ sont trois fonctions à variation bornée et où R, S sont deux constantes et où les intégrales sont prises au sens de Stieltjes.

Forme géométrique de la différentielle

14. Cette formule est susceptible d'une interprétation géométrique simple. Appelons λ, μ, ν les cosinus directeurs de la normale au plan T qui passe par la tangente à L_0 en un point P et par le déplacement Δx, Δy, Δz de ce point. Soient Δ_t, Δ_n les composantes de ce déplacement sur la tangente et suivant la normale à L_0 dans T et soient $\Delta\sigma_{yz}$, $\Delta\sigma_{zx}$, $\Delta\sigma_{xy}$ les projections sur les trois plans de coordonnées de l'aire du parallélogramme ayant pour cotés le déplacement $(\Delta x, \Delta y, \Delta z)$ de P et un segment unitaire sur la tangente à L_0 en P. On peut écrire la formule (29) en supposant que la variable initiale soit l'arc $s = AP$ de L_0. Seulement p, q, r, R, S ne seront pas nécessairement les mêmes fonctions, ni les mêmes constantes. On voit alors que *l'expression de la différentielle peut aussi s'écrire*

$$(30)\quad \int_0^l \Delta\sigma_{yz}(s)\,d[p_0(s)] + \int_0^l \Delta\sigma_{zx}(s)\,d[q_0(s)] + \int_0^l \Delta\sigma_{xy}(s)\,d[r_0(s)] + R_0\Delta_t(0) + S_0\Delta_t(l)$$

ou encore

$$(30\ bis)\quad \int_0^l \lambda(s)\,\Delta_n(s)\,d[p_0(s)] + \int_0^l \mu(s)\,\Delta_n(s)\,d[q_0(s)] + \int_0^l \nu(s)\,\Delta_n(s)\,d[r_0(s)] + R_0\Delta_t(0) + S_0\Delta_t(l).$$

Cette seconde expression montre que la valeur de la différentielle dépend des valeurs du déplacement normal Δ_n, mais qu'elle ne dépend du déplacement tangentiel Δ_t que par les valeurs de celui ci aux extrémités de la courbe. Ce résultat est d'autant plus intéressant que l'expression de notre différentielle s'applique à un déplacement infiniment petit *ou fini* et non pas seulement à la variation.

On remarquera l'analogie de l'expression (30) avec celle adoptée par M. Volterra pour la dérivée fonctionnelle. Elle s'en distingue d'abord par le fait que les Δ sont relatifs à des déplacements finis et non à des variations et en outre par l'introduction d'intégrales de Stieltjes, qui lui donne une plus grande généralité.

Remarque

15. Il y a lieu d'observer que les fonctions $p(t)$, $q(t)$, $r(t)$ ne sont assujetties à aucune condition analogue à (24). D'une manière précise, nous allons démontrer que *si* R, S *sont des constantes quelconques et si* $p(t)$, $q(t)$, $r(t)$ *sont des fonctions quelconques à variation bornée, on peut trouver des fonctions à variation bornée* $u(t)$, $v(t)$, $w(t)$ *satisfaisant à la condition* (24) *et telles que l'expression* (29) *soit de la forme* (23).* En effet, posons d'abord

$$(31)\quad \begin{aligned} u_3(t) &= \int_a^t z_0'(t)\,d[q(t)] - \int_a^t y_0'(t)\,d[r(t)], \\ v_3(t) &= \int_a^t x_0'(t)\,d[r(t)] - \int_a^t z_0'(t)\,d[p(t)], \\ w_3(t) &= \int_a^t y_0'(t)\,d[p(t)] - \int_a^t x_0'(t)\,d[q(t)]. \end{aligned}$$

Ces intégrales de Stieltjes ont un sens et on démontre—comme au § 12 pour la fonction $F(t)$—qu'elles définissent des fonctions $u_3(t)$, $v_3(t)$, $w_3(t)$ à variation bornée. Posons, pour $a < t < b$, $u(t) = u_3(t)$, $v(t) = v_3(t)$,

* Il faut remarquer que le résultat actuel serait tout aussi exact, si les coefficients de $\Delta x(a)$, $\Delta y(a)$, $\Delta z(a)$, $\Delta x(b)$, $\Delta y(b)$, $\Delta z(b)$ étaient quelconques et non respectivement proportionnels aux dérivées de x, y, z en a et b.

$w(t) = w_3(t)$. Alors non seulement $u(t)$, $v(t)$, $w(t)$ sont définies par ces formules pour $a < t < b$, mais encore aussi $u(a+0)$, $u(b-0)$, $\cdots$, $w(a+0)$, $w(b-0)$. Pour connaître partout u, v, w, il suffit donc de poser encore

$$u(a+0) - u(a) = Rx_0'(a) + u_3(a+0) - u_3(a), \tag{32}$$
$$u(b) - u(b-0) = Sx_0'(b) + u_3(b) - u_3(b-0)$$

et de même pour v, w. Nous remarquons alors que

$$\int_a^b \Delta x(t)\,\{d[u(t)] - d[u_3(t)]\} + \int_a^b \Delta y(t)\,\{d[v(t)] - d[v_3(t)]\} + \int_a^b \Delta z(t)\,\{d[w(t)] - d[w_3(t)]\}$$
$$= R[x_0'(a)\Delta x(a) + y_0'(a)\Delta y(a) + z_0'(a)\Delta z(a)] + S[x_0'(b)\Delta x(b) + y_0'(b)\Delta y(b) + z_0'(b)\Delta z(b)].$$

Pour prouver que l'expression (29) peut se mettre sous la forme (23), il nous suffit donc de démontrer que trois expressions de la forme

$$\int_a^b \Delta x(t)\,d[u_3(t)] - \int_a^b \Delta x(t)\,z_0'(t)\,d[q(t)] + \int_a^b \Delta x(t)\,y_0'(t)\,d[r(t)] \tag{33}$$

sont nulles. Or cette dernière est la limite d'une somme telle que

$$\sum_i \Delta x(\xi_i)\left\{[u_3(t_i) - u_3(t_{i-1})] - z_0'(\xi_i)\int_{t_{i-1}}^{t_i} d[q(t)] + y_0(\xi_i)\int_{t_{i-1}}^{t_i} d[r(t)]\right\} \tag{34}$$

qui est égale d'après (31) à

$$\sum_i \Delta x(\xi_i)\left\{\int_{t_{i-1}}^{t_i} [z_0'(t) - z_0'(\xi_i)]\,d[q(t)] - \int_{t_{i-1}}^{t_i} [y_0'(t) - y_0'(\xi_i)]\,d[r(t)]\right\}$$

lequel est en valeur absolue inférieur à

$$N\omega(\epsilon)[Q(b) + R(b)],$$

$Q(b)$ et $R(b)$ étant les variations totales de $q(t)$ et $r(t)$ dans (a, b). Quand le plus grand, ϵ, des intervalles $(t_i - t_{i-1})$ tend vers zéro, cette expression et par suite l'expression (34) tendent vers zéro. Les trois expressions telles que (33) sont bien nulles.

Reste à prouver que les fonctions u, v, w que nous venons de définir satisfont bien à la condition (24) sauf peut-être en un ensemble dénombrable de points. Il suffit évidemment de la prouver pour u_3, v_3, w_3, qui ne diffèrent de u, v, w qu'en deux points. Or la démonstration de la nullité de (33) s'applique évidemment au cas où la limite supérieure de l'intervalle d'intégration

serait t et non b. Elle s'applique quelles que soient les fonctions continues $\Delta x(t)$, $\Delta y(t)$, $\Delta z(t)$. On arrive alors au résultat énoncé en remplaçant dans l'égalité obtenue b par t et Δx, Δy, Δz par x_0', y_0', z_0'.

16. Mais à côté de l'avantage—présenté par la forme (29) de la différentielle sur la forme (23)—que nous venons de mentionner, il faut placer plusieurs inconvénients. D'abord nous avons dû supposer que la courbe L_0 est une courbe rectifiable à tangente continue, tandis que la forme (23) s'applique à une courbe continue quelconque. Et surtout le système des fonctions p, q, r est largement indéterminé pour la différentielle d'une fonction de ligne donnée. Ainsi l'expression (29) garde évidemment la même valeur quand on remplace p, q, r par $p+\rho x$, $q+\rho y$, $r+\rho z$, où ρ désigne une constante quelconque.

Parties régulières et irrégulières de la différentielle

17. Les deux formes précédentes (23) et (29) de la différentielle présentent celle-ci comme un bloc. Il y a avantage àemployer une autre forme qui manifeste l'existence dans la différentielle de parties plus ou moins régulières, plus ou moins simples.

Nous allons donc transformer le résultat précédent en utilisant une décomposition de l'intégrale de Stieltjes en trois parties de complications différentes. C'est cette décomposition à laquelle j'ai fait allusion au § 7 et dont j'ai démontré l'existence et l'unicité au Congrès des Sociétés Savantes tenu à Grenoble en Mai, 1913. J'y ai montré que toute intégrale de Stieltjes

$$\int_a^b x(t)\,d[u(t)]$$

peut se mettre sous la forme suivante

$$\int_a^b x(t)\,d[u(t)] = \sum_n A_n x(t_n) + \int_a^b \alpha(t)\,x(t)\,dt + \int_a^b x(t)\,d[\lambda(t)].$$

Pour former cette expression, on forme la " fonction des sauts " de $u(t)$

$$(35)\quad u_1(t) = \sum_{a<t_n\leqq t} [u(t_n) - u(t_n - 0)] + \sum_{a\leqq t_n<t} [u(t_n + 0) - u(t_n)],$$

où les t_n sont les points de discontinuité de $u(t)$. Puis, on pose $u_2(t) = u(t) - u_1(t)$; $u_2(t)$ est une fonction continue à variation bornée; elle a donc une dérivée presque partout; on prendra pour $\alpha(t)$ la valeur de cette dérivée là où elle existe et une valeur finie quelconque, par exemple zéro, ailleurs. La fonction $\alpha(t)$ sera donc sommable. Enfin, on posera

$$\lambda(t) = u_2(t) - \int_a^t \alpha(t)\,dt;$$

la fonction $\lambda(t)$ sera continue à variation bornée et elle aura presque partout une dérivée nulle. D'autre part, A_n est le saut de $u(t)$ au point t_n.

18. En opérant de même pour v et w, *on voit que la différentielle d'une fonction de ligne quelconque peut se mettre sous la forme suivante*

$$
\begin{aligned}
T_{\Delta x,\,\Delta y,\,\Delta z} = &\int_a^b \{\,\Delta x(t)\,\alpha(t) + \Delta y(t)\,\beta(t) + \Delta z(t)\,\gamma(t)\,\}\,dt \\
&+ \sum_n \{\,A_n\,\Delta x(t_n) + B_n\,\Delta y(t_n) + C_n\,\Delta z(t_n)\,\} \\
&+ \Big[\int_a^b \Delta x(t)\,d[\lambda(t)] + \int_a^b \Delta y(t)\,d[\mu(t)] \\
&\qquad + \int_a^b \Delta z(t)\,d[\nu(t)]\Big].
\end{aligned}
\tag{36}
$$

On voit de suite que cette formule diffère des formules ordinaires du Calcul des Variations non seulement en ce que les fonctions x, y, z, y entrent par leur accroissement et non par leur "variation" mais en ce qu'en outre elle comprend d'autres termes en dehors de l'intégrale ordinaire et des termes aux limites.

19. Nous devons remarquer que les trois termes du second membre de l'expression précédente peuvent aussi s'écrire, en posant

$$u_0(t) = \int_a^t \alpha(t)\,dt$$

et en raisonnant pour v, w comme pour $u(t)$, sous la forme d'intégrales de Stieltjes:

$$
\begin{aligned}
&\int_a^b \{\,\Delta x(t)\,d[u_0(t)] + \Delta y(t)\,d[v_0(t)] + \Delta z(t)\,d[w_0(t)]\},\\
&\int_a^b \{\,\Delta x(t)\,d[u_1(t)] + \Delta y(t)\,d[v_1(t)] + \Delta z(t)\,d[w_1(t)]\},\\
&\int_a^b \{\,\Delta x(t)\,d[\lambda(t)] + \Delta y(t)\,d[\mu(t)] + \Delta z(t)\,d[\nu(t)]\}.
\end{aligned}
\tag{37}
$$

Nous allons montrer que *la condition* (23) *applicable à la somme des trois termes précédent, s'applique aussi à chacun d'eux séparément, et qu'elle prend une forme particulièrement simple.*

20. On a en effet d'après (24)

$$K(t) = \text{une certaine constante } k,$$

sauf peut-être en un ensemble dénombrable de points. Dès lors on a *partout*

$$K(t \pm 0) = k, \tag{38}$$

et, d'après (22),

$$
\begin{aligned}
0 = x_0'(t)\,[u(t+0) - u(t-0)] + y_0'(t)\,[v(t+0) - v(t-0)] \\
+ z_0'(t)\,[w(t+0) - w(t-0]).
\end{aligned}
$$

En écrivant cette égalité pour les points t_n—pour les autres, elle est évidente —*on aura*

$$A_n x_0'(t_n) + B_n y_0'(t_n) + C_n z_0'(t_n) = 0 \quad (n = 1, 2, 3, \cdots), \tag{39}$$

pourvu toutefois que t_n soit distinct de a et de b.

Or on a d'après (35)

$$\int_a^t x_0' \, d[u_1(t)] = x_0'(t)[u(t) - u(t-0)] + \sum_{a<t_n<t} A_n x_0'(t_n) + x_0'(a)[u(a+0) - u(a)]$$

et de même d'après (22), (38), et (39)

$$\int_a^t [x_0' \, du_1 + y_0' \, dv_1 + z_0' \, dw_1] = K(t) - K(a) = K(t), \tag{40}$$

puisque $K(a)$ est évidemment nul. On voit donc que la condition analogue à (24) est bien vérifiée pour le second terme de (37). De plus, *elle résulte des relations simples* (39) qui la remplacent.

21. Mais, en remplaçant, dans (40), $K(t)$ par son expression, il reste

$$\int_a^t [x_0' \, du_2 + y_0' \, dv_2 + z_0' \, dw_2] = 0, \tag{41}$$

et ceci pour toute valeur de t. Or soit $U_2(t)$ la variation totale de $u_2(t)$ de a à t. Les fonctions $u_2(t)$ et $U_2(t)$ sont à variation bornée et continues. Elles sont donc dérivables presque partout. Si t_0 est un point où elles sont toutes deux dérivables et si l'on écrit la formule analogue à (17) divisée par $t' - t_0$

$$\left| \frac{\int_{t_0}^{t'} x_0'(t) \, d[u_2(t)]}{t' - t_0} - x_0'(t_0) \frac{u_2(t') - u_2(t_0)}{t' - t_0} \right| < \Omega(\epsilon) \left| \frac{U_2(t') - U_2(t_0)}{t' - t_0} \right|,$$

on voit en passant à la limite que la fonction

$$\int_a^t x_0'(t) \, d[u_2(t)]$$

a une dérivée au point t_0 égale à $x_0'(t_0)$ multipliée par $d[u_2(t_0)]/dt_0$. En opérant de même pour v_2 et w_2 et se souvenant de la définition de α, β, γ, on voit que le premier membre de (41) a presque partout une dérivée égale à $x_0'(t)\alpha(t) + y_0'(t)\beta(t) + z_0'(t)\gamma(t)$. Comme le second membre est nul, *on a en définitive presque partout*

$$x_0'(t)\alpha(t) + y_0'(t)\beta(t) + z_0'(t)\gamma(t) = 0. \tag{42}$$

On peut même supposer que cette égalité a lieu partout en changeant au besoin les valeurs de $\alpha(t)$, $\beta(t)$, $\gamma(t)$ sur un ensemble de points de mesure nulle, ce qui n'altérera pas la formule (36). Si par exemple u_2, v_2, w_2 ont des dérivées continues partout, α, β, γ seront égaux à ces dérivées et l'équation (42) ayant lieu presque partout aura lieu nécessairement partout. D'ailleurs si l'on intégre (42) de a à t, en remarquant que

$$u_0(t) = \int_a^t \alpha(t)\,dt, \qquad v_0(t) = \int_a^t \beta(t)\,dt, \qquad w_0(t) = \int_a^t \gamma(t)\,dt,$$

on voit qu'on aura partout

$$(43) \qquad \int_a^t x_0'(t)\,d[u_0(t)] + y_0'(t)\,d[v_0(t)] + z_0'(t)\,d[w_0(t)] = 0.$$

Donc la condition (24) s'applique aussi au premier terme de (37). *Et elle se traduit par la condition plus simple* (42). Enfin en retranchant (43) de (41), on voit que la condition (24) s'applique aussi au troisième terme de (37), car on obtient partout

$$(44) \qquad \int_a^t x_0'(t)\,d[\lambda(t)] + y_0'(t)\,d[\mu(t)] + z_0'(t)\,d[\nu(t)] = 0.$$

Autre forme de la différentielle

22. L'expression (36) de la différentielle de U_L contient des constantes et des fonctions liées par les relations (39), (42) et (44). On peut alors en déduire une nouvelle forme de la différentielle où l'on a tenu compte d'avance de ces relations. Cette forme sera à la forme (36) ce qu'est la forme (29) à la forme (23). Nous ferons donc la même hypothèse que pour (29), à savoir que L_0 est rectifiable et à tangente continue.

Étant donnée la relation (39), on peut toujours choisir des constantes P_n, Q_n, R_n, telles que l'on ait

$$(45) \qquad \begin{aligned} A_n &= Q_n z_0'(t_n) - R_n y_0'(t_n), \qquad B_n = R_n x_0'(t_n) - P_n z_0'(t_n), \\ C_n &= P_n y_0'(t_n) - Q_n x_0'(t_n). \end{aligned}$$

Par suite le second terme de la différentielle (36) devient

$$(46) \qquad \begin{aligned} \sum_n \{ & P_n [y_0'(t_n)\,\Delta z(t_n) - z_0'(t_n)\,\Delta y(t_n)] \\ & + Q_n [z_0'(t_n)\,\Delta x(t_n) - x_0'(t_n)\,\Delta z(t_n)] \\ & + R_n [x_0'(t_n)\,\Delta y(t_n) - y_0'(t_n)\,\Delta x(t_n)]\}, \end{aligned}$$

où t_n est distinct de a et de b et *auquel il faut ajouter un terme de la forme*

$$(47)\quad A\Delta x(a) + B\Delta y(a) + C\Delta z(a) + A'\Delta x(b) + B'\Delta y(b) + C'\Delta z(b),$$

les relations (39) n'ayant pas lieu nécessairement aux extrémités.

De même, la relation (42) permet de choisir des fonctions sommables $P(t)$, $Q(t)$, $R(t)$ telles que l'on ait presque partout

$$\alpha(t) = Q(t)\,z_0'(t) - R(t)\,y_0'(t), \qquad \beta(t) = R(t)\,x_0'(t) - P(t)\,z_0'(t),$$

$$\gamma(t) = P(t)\,y_0'(t) - Q(t)\,x_0'(t).$$

Il nous suffit de prendre par exemple

$$P = \frac{\gamma y_0' - \beta z_0'}{x_0'^2 + y_0'^2 + z_0'^2}, \qquad Q = \frac{\alpha z_0' - \gamma x_0'}{x_0'^2 + y_0'^2 + z_0'^2}, \qquad R = \frac{\beta x_0' - \alpha y_0'}{x_0'^2 + y_0'^2 + z_0'^2}.$$

On peut supposer—en prenant au besoin pour variable t l'arc de la courbe—que $x'^2 + y'^2 + z'^2$ reste partout supérieur à un nombre $h > 0$, de sorte que P, Q, R sont bien sommables. Alors le *premier terme de la différentielle* (26) *peut s'écrire*

$$\int_a^b \{P(t)\,[y_0'(t)\Delta z(t) - z_0'(t)\Delta y(t)] + Q(t)\,[z_0'(t)\Delta x(t) - x_0'(t)\Delta z(t)] + R(t)\,[x_0'(t)\,\Delta y(t) - y_0'(t)\,\Delta x(t)]\}\,dt.$$

On retrouve ainsi l'expression qui constitue,—après remplacement des accroissements de x, y, z par leurs "variations,"—la dérivée fonctionnelle au sens de M. Volterra. Mais nous voyons qu'il s'y ajoute des termes d'espèce différentes, les termes (46), (47) et ceux qui résultent de la transformation, que nous allons opérer, du troisième terme de (36). Il nous suffit d'appliquer à ce troisième terme la transformation même qui a été appliquée dans le § 13 à la différentielle tout entière prise sous la forme (23). Seulement, nous remarquons que la fonction qui remplace $K(t)$ est d'après (44) *partout* égale à zéro. De sorte que l'expression qui correspond au δ du § 13 est cette fois nulle. D'autre part, les fonctions qui correspondent à $p(t)$, $q(t)$, $r(t)$ s'obtenant en remplaçant dans les formules (25) $u(t)$, $v(t)$, $w(t)$ par $\lambda(t)$, $\mu(t)$, $\nu(t)$ non seulement seront à variation bornée mais encore seront continues et avec une dérivée nulle presque partout. La démonstration de ce fait est identique aux démonstrations analogues des §§ 12 et 21.

En résumé, on voit *que pour toute courbe L_0 rectifiable à tangente continue, on peut mettre la différentielle d'une fonction de ligne U_L sous la forme*

$$
\begin{aligned}
&\int_a^b \{ (P(t)\,[\,y_0'(t)\,\Delta z(t) - z_0'(t)\,\Delta y(t)\,] \\
&+ Q(t)\,[\,z_0'(t)\,\Delta x(t) - x_0'(t)\,\Delta z(t)\,] \\
&+ R(t)\,[x_0'(t)\,\Delta y(t) - y_0'(t)\,\Delta x(t)\,]\}\,dt \\
&+ \int_a^b [\,y_0'(t)\,\Delta z(t) - z_0'(t)\,\Delta y(t)\,]\,d[\,X(t)\,] \\
&+ \int_a^b [\,z_0'(t)\,\Delta x(t) - x_0'(t)\,\Delta z(t)\,]\,d[\,Y(t)\,] \\
&+ \int_a^b [\,x_0'(t)\,\Delta y(t) - y_0'(t)\,\Delta x(t)\,]\,d[\,Z(t)\,] \\
&+ \sum_n \{\, P_n\,[\,y_0'(t_n)\,\Delta z(t_n) - z_0'(t_n)\,\Delta y(t_n)\,] \\
&+ Q_n\,[\,z_0'(t_n)\,\Delta x(t_n) - x_0'(t_n)\,\Delta z(t_n)\,] \\
&+ R_n\,[\,x_0'(t_n)\,\Delta y(t_n) - y_0'(t_n)\,\Delta x(t_n)\,]\} \\
&+ A\Delta x(a) + B\Delta y(a) + C\Delta z(a) \\
&+ A'\Delta x(b) + B'\Delta y(b) + C'\Delta z(b),
\end{aligned}
\tag{48}
$$

où les A, B, C, A', B', C', P_n, Q_n, R_n sont des constantes quelconques, où les t_n sont des points fixes distincts des extrémités, où $P(t)$, $Q(t)$, $R(t)$ sont des fonctions sommables dans (a, b) et où $X(t)$, $Y(t)$, $Z(t)$ sont des fonctions continues à variation bornée et qui admettent presque partout une dérivée première nulle.

Transformation de la différentielle en intégrale de M. Lebesgue

23. On peut enfin transformer l'expression de la différentielle de façon à éliminer les intégrales de Stieltjes. Il suffit pour cela d'employer une méthode appliquée par M. Lebesgue* au cas d'une seule intégrale de Stieltjes. Je crois utile de développer cette méthode dans le cas un peu plus général actuel parce que M. Lebesgue n'est pas entré dans tous les détails et surtout parce que l'artifice analytique qu'il emploie parait assez subtil, tandis que dans le cas actuel il apparait sous une forme géométrique très naturelle.

Nous supposons seulement sur la courbe L_0 qu'elle est continue et nous partons de l'expression

$$
T_{\Delta x,\,\Delta y,\,\Delta z} = \int_a^b \Delta x(t)\,d[\,u(t)\,] + \int_a^b \Delta y(t)\,d[\,v(t)\,] + \int_a^b \Delta z(t)\,d[\,w(t)\,] \tag{49}
$$

*Comptes Rendus t. 150 (1910), p. 86–88.

de la différentielle. Il s'agit d'effectuer sur cet intégrale de Stieltjes de simples changements de variables qui la ramènent à une intégrale ordinaire ou plus exactement à une intégrale de Lebesgue. Il y a lieu de modifier—très légèrement—la démonstration de M. Lebesgue, qui s'applique à chaque intégrale séparément, de façon à employer le même changement de variable pour les trois intégrales.

Tout d'abord, d'après la définition des intégrales de Stieltjes, on sait qu'on peut modifier u, v, w en un ensemble dénombrable de valeurs de t (autres que a et b) sans changer la valeur de l'intégrale. *Nous pourrons donc supposer que* $u(t)$, $v(t)$, $w(t)$ *sont partout continues à droite,*—sauf peut-être pour $t = a$.

Nous allons—opérant un peu différemment de M. Lebesgue—employer deux changements de variable successifs sur t, le changement intermédiaire donnant déjà un résultat qui peut être utile. Soit Σk_n une série convergente quelconque à termes positifs. Posons

$$\varphi(a) = a, \qquad \varphi(t) = t + \sum_{t_n \leqq t} k_n \qquad (a < t \leqq b),$$

où les t_n sont les points de discontinuité des fonctions u, v, w. La fonction $\varphi(t)$ est une fonction croissante qui n'est discontinue qu'aux points t_n. Soit maintenant $\theta(T)$ la borne inférieure des valeurs de t telles que $\varphi(t) \geqq T$. En posant $B = \varphi(b)$, on voit que $\theta(T)$ est une fonction continue non décroissante pour $a \leqq T \leqq B$. De plus $\theta(T)$ prend la valeur t_n pour $\varphi(t_n - 0) \leqq T \leqq \varphi(t_n)$. Nous ferons d'abord dans la formule (49) le changement de variable $t = \theta(T)$, mais avec une petite modification.

Appelons $\mathbf{U}(T)$, $\mathbf{V}(T)$, $\mathbf{W}(T)$ les fonctions définies de la manière suivante. Nous remarquons que si t croît de a à b, $\varphi(t)$ croît de a à B, mais il est discontinu aux points t_n de sorte que

$$\varphi(t_n - 0) < \varphi(t_n) = \varphi(t_n + 0)$$

(si l'un des t_n est égal à a, on a au contraire $\varphi(a) < \varphi(a + 0)$. Quand T n'est pas dans un de ces intervalles exceptionnels nous poserons

$$\mathbf{U}(T) = u[\theta(T)], \qquad \mathbf{V}(T) = v[\theta(T)], \qquad \mathbf{W}(T) = w[\theta(T)].$$

Si au contraire, on a par exemple $\varphi(t_n - 0) \leqq T \leqq \varphi(t_n)$, on a $\theta(T) = t_n$, d'où $u[\theta(T)] = u(t_n)$. Dans ce cas, on prendra pour $\mathbf{U}(T)$ par exemple non $u(t_n)$ mais une fonction linéaire entre $T'_n = \varphi(t_n - 0)$ et $T_n = \varphi(t_n)$, prenant aux extrémités les valeurs $\mathbf{U}(T'_n) = u(t_n - 0)$, $\mathbf{U}(T_n) = u(t_n)$; et de même pour $\mathbf{V}(T)$, $\mathbf{W}(T)$.

On démontre alors facilement que la différentielle (49) peut s'écrire sous la forme d'une somme d'intégrales de Stieltjes

$$T_{\Delta x,\,\Delta y,\,\Delta z} = \int_a^B \Delta x\,[\theta(T)]\,d\,[\mathbf{U}(T)] + \int_a^B \Delta y\,[\theta(T)]\,d\,[\mathbf{V}(T)] + \int_a^B \Delta z\,[\theta(T)]\,d\,[\mathbf{W}(T)], \tag{50}$$

où $\mathbf{U}(T)$, $\mathbf{V}(T)$, $\mathbf{W}(T)$ *sont des fonctions continues et à variation bornée et où* $\theta(T)$ *est une fonction continue non décroissante, ces quatre fonctions restant indépendantes de* Δx, Δy, Δz.

On voit qu'on a ainsi simplifié d'une part l'expression (49) en remplaçant les fonctions u, v, w par des fonctions de même nature mais continues, et qu'on l'a compliqué d'autre part par l'introduction de la fonction $\theta(T)$, qui peut être constante dans certains intervalles.

24. Considérons $\mathbf{U}(T)$, $\mathbf{V}(T)$, $\mathbf{W}(T)$ comme les coordonnées d'un point variable; ce point décrit une courbe rectifiable continue Γ. *Mis sous forme géométrique, en passant d'une à trois coordonnées, l'artifice de M. Lebesgue consiste au fond à prendre comme nouvelle variable l'arc de cette courbe* et paraît maintenant s'imposer. Comme il pourrait arriver que $\mathbf{U}$, $\mathbf{V}$, $\mathbf{W}$ fussent en même temps constants dans un même intervalle, cette circonstance nous oblige à préciser la substitution adoptée. Appelons $\Psi(T)$ l'arc de la courbe Γ de a à T. $\Psi(T)$ est une fonction continue non décroissante. Soit $\Sigma = \Psi(B)$; c'est la longueur de la courbe Γ. Nous appellerons $\psi(\sigma)$ la borne inférieure des valeurs de T telles que $\Psi(T) \geqq \sigma$ et nous poserons

$$\Phi(\sigma) = \theta[\psi(\sigma)]; \qquad \mathbf{A}_1(\sigma) = \mathbf{U}[\psi(\sigma)]; \qquad \mathbf{B}_1(\sigma) = \mathbf{V}[\psi(\sigma)];$$

$$\mathbf{C}_1(\sigma) = \mathbf{W}[\psi(\sigma)].$$

On voit alors comme au paragraphe précédent, que la différentielle (49) peut s'écrire sous la forme

$$T_{\Delta x,\,\Delta y,\,\Delta z} = \int_0^\Sigma \Delta x\,[\Phi(\sigma)]\,d\,[\mathbf{A}_1(\sigma)] + \int_0^\Sigma \Delta y\,[\Phi(\sigma)]\,d\,[\mathbf{B}_1(\sigma)] + \int_0^\Sigma \Delta z\,[\Phi(\sigma)]\,d\,[\mathbf{C}_1(\sigma)]. \tag{51}$$

Or $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$ sont évidemment des fonctions continues à variation bornée et à nombres dérivés bornés. Elles ont donc presque partout chacune une dérivée, et même il existe un ensemble E de valeurs de σ de mesure Σ dans l'intervalle $(0, \Sigma)$ où les dérivées de $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$ existent et où la somme de leurs carrés est égale à 1. Définissons alors des fonctions $\mathbf{A}(\sigma)$, $\mathbf{B}(\sigma)$, $\mathbf{C}(\sigma)$ égales aux dérivées respectives de $\mathbf{A}_1$, $\mathbf{B}_1$, $\mathbf{C}_1$ en chaque point de E et égales à $1/\sqrt{3}$

partout ailleurs. On aura alors quel que soit σ dans l'intervalle $(0, \sigma)$

$$(52)\quad \mathbf{A}_1(\sigma) = \int_0^\sigma \mathbf{A}(\sigma)\,d\sigma,\ \mathbf{B}_1(\sigma) = \int_0^\sigma \mathbf{B}(\sigma)\,d\sigma,\ \mathbf{C}_1(\sigma) = \int_0^\sigma \mathbf{C}(\sigma)\,d\sigma,$$
$$[\mathbf{A}(\sigma)]^2 + [\mathbf{B}(\sigma)]^2 + [\mathbf{C}(\sigma)]^2 = 1.$$

Les fonctions $\mathbf{A}, \mathbf{B}, \mathbf{C}$ étant sommables, $\Delta x[\Phi(\sigma)], \Delta y[\Phi(\sigma)], \Delta z[\Phi(\sigma)]$ étant mesurables et bornées, l'intégrale

$$(53)\quad \int_0^\Sigma \{\Delta x[\Phi(\sigma)]\,\mathbf{A}(\sigma) + \Delta y[\Phi(\sigma)]\,\mathbf{B}(\sigma) + \Delta z[\Phi(\sigma)]\,\mathbf{C}(\sigma)\}\,d\sigma$$

a un sens. Nous allons montrer qu'elle est égale au second membre de (51). Nous considérerons dans ce but les fonctions $\mathbf{F}(\sigma)$, $\mathbf{G}(\sigma)$ de σ obtenues en remplaçant dans (51) et (53) les limites supérieures d'intégration par σ. La seconde a comme on sait l'élément d'intégration pour dérivée presque partout. Un raisonnement analogue à celui de § 21 démontre que la première a la même dérivée en tout point où $\Delta x[\Phi(\sigma)], \Delta y[\Phi(\sigma)], \Delta z[\Phi(\sigma)]$ sont continues et où $\mathbf{A}_1(\sigma), \mathbf{B}_1(\sigma), \mathbf{C}_1(\sigma)$ et leurs variations totales ont une dérivée, c'est à dire aussi, presque partout. Ainsi $[\mathbf{F}(\sigma) - \mathbf{G}(\sigma)]$ a une dérivée nulle presque partout. On voit facilement que les nombres dérivés à droite de $\mathbf{F}(\sigma)$ et $\mathbf{G}(\sigma)$ sont bornés; donc aussi ceux de $\mathbf{F}(\sigma) - \mathbf{G}(\sigma)$. Par suite $\mathbf{F}(\sigma) - \mathbf{G}(\sigma)$ est l'intégrale indéfinie d'un de ces nombres dérivés là où il existe et comme il est nul presque partout, $\mathbf{F}(\sigma) - \mathbf{G}(\sigma)$ est une constante. Cette constante est nulle pour $\sigma = 0$. On a bien enfin $\mathbf{F}(\sigma) = \mathbf{G}(\sigma)$. En définitive, *la différentielle d'une fonction de ligne peut se mettre pour une ligne continue quelconque sous la forme d'une intégrale de Lebesgue:*

$$(54)\quad T_{\Delta x,\,\Delta y,\,\Delta z} = \int_0^\Sigma \{\Delta x[\Phi(\sigma)]\,\mathbf{A}(\sigma) + \Delta y[\Phi(\sigma)]\,\mathbf{B}(\sigma) + \Delta z[\Phi(\sigma)]\,\mathbf{C}(\sigma)\}\,d\sigma,$$

où $\Phi(\sigma)$ est une fonction non décroissante, où $\mathbf{A}(\sigma), \mathbf{B}(\sigma), \mathbf{C}(\sigma)$ sont des fonctions sommables bornées, et où ces quatre fonctions sont indépendantes de $\Delta x, \Delta y, \Delta z$.

25. Seulement, il faut remarquer qu'on n'obtiendra pas une forme admissible de la différentielle en choisissant pour $\mathbf{A}, \mathbf{B}, \mathbf{C}$ des fonctions sommables bornées arbitraires. Ces fonctions dérivent en effet, par une suite d'opérations déterminées, des fonctions u, v, w lesquelles doivent vérifier une certaine condition. Lorsque la courbe L_0 est rectifiable et à tangente continue, cette condition s'exprime sous la forme (24). Il n'est pas nécessaire d'obtenir la relation correspondante entre $\mathbf{A}, \mathbf{B}, \mathbf{C}$ pour transformer l'expression (55) sous une forme ne faisant intervenir que des fonctions sommables et bornées arbitraires.

Il nous suffit en effet d'utiliser le procédé qui nous a permis de passer de la forme (49) à la forme (54) et de l'appliquer à la forme explicite (29).

On voit alors que *pour une courbe rectifiable à tangente continue la différentielle peut se mettre sous la forme de la somme d'un terme fini et d'une intégrale de Lebesgue*

$$\begin{aligned} T_{\Delta x,\,\Delta y,\,\Delta z} = \int_0^H \{ & [\, y_0'[\Theta(\sigma)]\,\Delta z[\Theta(\sigma)] - z_0'[\Theta(\sigma)]\,\Delta y[\Theta(\sigma)]\,]\,P(\sigma) \\ & + [\, z_0'[\Theta(\sigma)]\,\Delta x[\Theta(\sigma)] - x_0'[\Theta(\sigma)]\,\Delta z[\Theta(\sigma)]\,]\,Q(\sigma) \\ & + [x_0'[\Theta(\sigma)]\,\Delta y[\Theta(\sigma)] - y_0'[\Theta(\sigma)]\,\Delta x[\Theta(\sigma)]\,]\,R(\sigma)\,\}\,d\sigma \\ & + R[\, x_0'(a)\,\Delta x(a) + y_0'(a)\,\Delta y(a) + z_0'(a)\,\Delta z(a)\,] \\ & + S[\, x_0'(b)\,\Delta x(b) + y_0'(b)\,\Delta y(b) + z_0'(b)\,\Delta z(b)\,] \end{aligned}$$

où $\Theta(\sigma)$ est une fonction non décroissante, $P(\sigma)$, $Q(\sigma)$, $R(\sigma)$ sont des fonctions sommables bornées, R, S, H sont des constantes, et où ces fonctions et ces constantes sont indépendantes de Δx, Δy, Δz.

Poitiers,
le 16 *Mai* 1913.

A TYPE OF PRIMITIVE ALGEBRA*

BY

J. H. M. WEDDERBURN

In a recent paper,† L. E. Dickson has discussed the linear associative algebra, A, defined by the relations

$$xy = y\theta(x), \qquad y^n = g,$$

where $\theta(x)$ is a polynomial in x which is rational in the field F in which A is defined. In this paper Dickson shows for $n = 2$ and $n = 3$ that, when θ and g are properly chosen, A is primitive, i. e., every element in it with the exception of zero has an inverse: A is in fact in the case $n = 2$ a direct generalization of quaternions.

So far as I am aware no algebra other than these two and fields has been proved to be primitive; hence it is of considerable interest to find that for any value of n, θ and g can be so chosen as to make Dickson's algebra primitive.

In discussing primitive algebras we may without any real loss of generality assume that any element commutative with every other element of the algebra is a scalar, i. e., an element of the field F. For, all such elements generate a commutative subalgebra B and, as in the theory of groups, we can find a complex C for which

$$A = BC = CB.$$

We can therefore regard A as an algebra whose coefficients lie in the field F extended by the elements of the commutative primitive algebra B and, in this algebra, scalars are the only elements commutative with every other element.

A primitive algebra will be called *normal* when reduced in this manner.

It follows from the general theory of linear associative algebras‡ that a normal primitive algebra is of order n^2 and that, when the field is sufficiently extended, it is equivalent to the simple matric algebra e_{pq} $(p, q = 1, 2, \cdots, n)$, so that its identical equation is of degree n. We shall now investigate the consequences that result from the assumption that the normal primitive

* Presented to the Society, December 31, 1913.

† These Transactions, vol. 15 (1914), pp. 31–46.

‡ See, for instance, my paper in the Proceedings of the London Mathematical Society, ser. 2, vol. 6 (1907), pp. 77–118, where references are given.

algebra A contains an element for which the group of the corresponding identical equation is the cyclic group of order n.

1. Let A be a normal primitive algebra of order n^2 and let x be an element of it for which the group of the identical equation $f(x) = 0$ is cyclic.

If $\xi_1, \xi_2, \cdots, \xi_n$ are the roots of x, the corresponding primitive idempotent elements are

$$e_r \equiv e_{rr} = \frac{(x-\xi_1)\cdots(x-\xi_{r-1})(x-\xi_{r+1})\cdots(x-\xi_n)}{(\xi_r-\xi_1)\cdots(\xi_r-\xi_{r-1})(\xi_r-\xi_{r+1})\cdots(\xi_r-\xi_n)} \qquad (r=1,2,\cdots,n)$$

and* when F is extended by the roots of x, A contains a matric† algebra (e_{pq}) which does not reduce to $e_1, e_2, \cdots, e_n$, since then x would be commutative with every element of A. We may therefore suppose $e_{12} \neq 0$, so that there is an element y_1, rational in F, for which $e_1 y_1 e_2$ is not zero. The conjugates of $e_1 y_1 e_2$ in F are $e_r y_1 e_{r+1}$ $(r = 1, 2, \cdots, n)$, none of which can therefore be zero.

If we set

$$e_r y_1 e_{r+1} = \eta_{r,r+1} e_{r,r+1},$$

the element

$$y = \sum_{r=1}^{n} \eta_{r,r+1} e_{r,r+1}$$

is rational since it is the sum of the conjugates of $e_1 y_1 e_2$; its nth power is evidently the scalar $\eta_{12}\eta_{23}\cdots\eta_{n1} = g$, say, which is therefore rational. Since

$$x = \sum_{r=1}^{n} \xi_r e_r,$$

$$xy = \sum_{r=1}^{n} \xi_r \eta_{r,r+1} e_{r,r+1} = yx_1,$$

where

$$x_1 = \sum_{r=1}^{n} \xi_{r-1} e_r.$$

The element x_1 is commutative with x and is therefore a rational polynomial in x, say $\theta(x)$, since the identical equation, $f(x) = 0$, has no repeated roots. Evidently x and y generate A when expressed in its matric form and therefore also in its rational form.

We have therefore the following

THEOREM. *A normal primitive algebra which contains an element x whose group is cyclic is generated by x and an element y which satisfies the relations*

* See Proceedings of the London Mathematical Society, *l. c.*, p. 97.

† A matric algebra is one for which a basis (e_{pq}) can be chosen for which $e_{pq} e_{rs} = 0$, $q \neq r$, $e_{pq} e_{qr} = e_{pr}$, $\Sigma e_{rr} = 1$. Some of the elements e_{pq} may be zero, but if so the algebra is reducible.

$$xy = y\theta(x), \qquad y^n = g,$$

where $\theta(x)$ and g are rational in the field of the coefficients and y^n is the first power of y which is commutative with x.*

Conversely if x and y satisfy these conditions, the group of x is the cyclic group of order n. This may be shown as follows.

Since

$$xy = y\theta(x),$$

therefore

$$xy^r = y^r\theta^r(x),$$

where $\theta^r(x)$ denotes the function θ iterated r times. If $\theta^r(x) = x$, then y^r is commutative with x, which is contrary to the given conditions unless $r = n$.

The matric form of y can be considerably simplified by choosing the units e_{pq} in such a way that $\eta_{r,\,r+1}\,e_{r,\,r+1}$ is replaced by $e_{r,\,r+1}$ for $r = 1, 2, \cdots, n-1$, e_{n1} being chosen so that

$$e_{12}\,e_{23}\cdots e_{n-1,\,n}\,e_{n1} = e_{11}.$$

This change in the basis of (e_{pq}) leaves e_{rr} $(e = 1, 2, \cdots n)$ unaltered so that x has the same form as before while y becomes

$$y = e_{12} + e_{23} + \cdots + e_{n-1,\,n} + ge_{n1}.$$

It is easily shown from the forms given above for x and y that any element, z, of A can be expressed uniquely in the form

$$y^{n-1}h_0 + y^{n-2}h_1 + \cdots + yh_{n-2} + h_{n-1},$$

where h_r $(r = 0, 1, \cdots, n-1)$ are rational polynomials in x.

Now

$$y^r = e_{1,\,r+1} + e_{2,\,r+2} + \cdots + e_{n-r,\,n} + g\,(e_{n-r+1,\,1} + \cdots + e_{nr})$$

and

$$y^r x^s = \xi^s_{r+1}\,e_{1,\,r+1} + \xi^s_{r+2}\,e_{1,\,r+2} + \cdots,$$

so that in the matric form of any element of A all the coefficients to the left of the principal diagonal are multiplied by g, and if any particular coefficient is zero all the coefficients in the same diagonal are zero.

2. We shall now determine the conditions which g must satisfy in order that an algebra of Dickson's type shall be primitive.

Let us consider an element of the form

$$z = y^r + y^{r-1}h_1 + \cdots + yh_{r-1} + h_r \qquad (h_r \neq 0)$$

*We may evidently modify y so that g is an algebraic integer of the field.

and let $k_1, k_2, \cdots, k_{n-r}$ be polynomials in x to be determined later, then

$$(y^{n-r} + y^{n-r-1} k_1 + \cdots + k_{n-r})z = g + y^{n-1} h_1 + y^{n-2} h_2 + \cdots + y^{n-r} h_r$$
$$+ y^{n-1} y^{-r} k_1 y^r + y^{n-2} y^{-r+1} k_1 y^{r-1} h_1 + \cdots + y^{n-2} y^{-r} k_2 y^r + \cdots + \cdots .$$

If we put $k_1 = - y^r h_1 y^{-r}$, which is a known rational polynomial in x, the coefficient of y^{n-1} is zero. Similarly the coefficient of y^{n-2} vanishes if

$$k_2 = - yk_1 y^{r-1} h_1 y^{-r} - y^r h_2 y^{-r},$$

which is also a known polynomial in x, and so on till the coefficient of y^r is reached, after which the process terminates. We can therefore in general determine the k's so that the product commences with a term $y^{r-s} h(x)$ $(s < r)$ and, after multiplying on the right by $1/h(x)$, which exists if $h \neq 0$, we have a number which begins with the term y^{r-s}. If this number has an inverse, z will also have one, since, by a well-known theorem on matrices, if a product has an inverse, the same is true of each factor. We have therefore only two types of elements to consider: first, those which may be reduced to the form $y + \varphi(x)$ by repeated applications of the process given above; and second elements for which at some stage the product is independent of y.

If $y + \varphi(x)$ is expressed in its matric form, its determinant is easily seen to be

$$\varphi(\xi_1)\, \varphi(\xi_2) \cdots \varphi(\xi_n) - g,$$

so that, if g is not the norm of any rational function of ξ, any element of this type has an inverse.

Suppose now that z is an element of the second of the two types mentioned above, so that there exists an element

$$z_1 = y^{n-r} + y^{n-r-1} k_1 + \cdots + k_{n-r},$$

such that $z_1 z$ is independent of y. The determination of the coefficients $k_1, \cdots, k_{n-r}$ is wholly independent of g, as the transform of any polynomial in x by a power of y depends solely on $\theta(x)$. We may therefore set

$$z_1 z = g + k,$$

where k is independent of g, which may be regarded as a variable scalar.

If k is not itself a scalar, $g + k$ certainly has an inverse and therefore z has also. We may assume therefore that k is a scalar, so that the matrix corresponding to z_1 differs from the adjoint of the matrix corresponding to z merely by a scalar factor which must be an integral function of g since z_1, z and $g + k$ are all integral in g. Since the determinant of the adjoint of a matrix is a power of the determinant of that matrix, the determinant of z,

say $|z|$, must be a power of $g+k$, and as $|z|$ is obviously of the rth degree in g, beginning with the term $(-1)^{n-r}g^r$, we have

$$|z| = (-1)^{n-r}(g+k)^r.$$

But, when $g=0$, $|z|$ becomes the norm of h_r so that

$$(-1)^{n-r}k^r = N(h_r).$$

Unless the scalar $g+k$ is zero, z has an inverse. In the contrary case, $g^r = N(-h_r)$. Hence, *if no power of g less than the nth is the norm of a rational polynomial in x, every element of the algebra, except zero, has an inverse.**

*The existence of such rational numbers g follows from Satz 33 of Hilbert's *Bericht*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 4 (1897), p. 198.

PROPERTIES OF SURFACES WHOSE ASYMPTOTIC CURVES BELONG TO LINEAR COMPLEXES*

BY

CHARLES T. SULLIVAN

INTRODUCTION

In this paper a study is made of the geometrical properties of surfaces whose asymptotic curves belong to linear complexes. The treatment is based on the methods developed by E. J. Wilczynski in his book on projective differential geometry† and a series of memoirs published in these Transactions.‡

So far as I am aware, the only papers on this subject are a note by Sophus Lie§ and a thesis by Arnold Peter.‖ In his thesis, Peter gives the analytical details of Lie's note and establishes the theorem that "*The determination of the surfaces can be reduced to quadratures.*"

One of the theorems established below, namely, "The ruled surfaces of the problem have straight line directrices" is due to Peter. The method of proof employed here, however, is entirely different from that used by Peter and is, moreover, essentially connected with the subsequent study of non-ruled surfaces.

In my work, as well as in that of Peter, a certain quadric surface plays an important rôle. From my point of view this quadric is the locus of the directrices of the osculating ruled surfaces associated with the two families of

* Presented to the Society (Chicago), April, 1912.

† E. J. Wilczynski, *Projective Differential Geometry of Curves and Ruled Surfaces.* B. G. Teubner, Leipzig, 1906. We shall hereafter refer to this book as W.

‡ Vol. 8 (1907), pp. 223–260; vol. 9 (1908), pp. 79–120, 293–315. We shall hereafter refer to these as M_1, M_2, M_3.

§ Sophus Lie, Christiania Videnskabsselskabs Forhandlinger, 1882, Nr. 21.

‖ Arnold Peter, *Die Flächen deren Haupttangentenkurven linearen Komplexen angehören* Leipzig, Dissertation, 1895. Since the presentation of this paper to the Society, some points of its subject matter have been touched upon in two other papers, viz., one by Enrico Bompiani, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912); the other by Corrado Segre, Reale Accademia delle Scienze di Torino, vol. 49 (1913). I have recently learned that a certain aspect of the subject of this paper was treated by M. Keraval, Bulletin de la Societé Mathématique de France, vol. 39 (1911). The methods used in these papers are entirely unlike those employed here, and the results are only distantly related to those of this paper.

asymptotic curves. If the asymptotic curves of only one of the two families belong to linear complexes, this quadric is replaced by a directrix-ruled surface of a higher order. On the other hand, the quadric introduced by Peter is characteristic of the canonical differential equations of the surfaces. However there is nothing in Peter's thesis concerning the geometrical significance of this quadric, nor is there anything analogous to my discussion of the osculating ruled surfaces and the directrix-ruled surface. All of the other results in the two papers are entirely unrelated and therefore require no further comparison.* The two principal results obtained in this paper are:

A geometric construction for the surfaces of the problem (§ 4), and a normal form for their defining equations (§ 3). In the normal form of these equations the coefficients, and therefore all the invariants of the surfaces, are given explicitly as functions of the two parameters u and v of the asymptotic curves.

I take this opportunity of expressing my indebtedness to Professor Wilczynski for advice generously given me from time to time during the preparation of this paper.

§ 1. Theorems concerning ruled surfaces whose asymptotic curves belong to linear complexes

Any non-developable ruled surface may be defined by a system of differential equations of the form†

$$\begin{aligned} y'' + p_{11}\,y' + p_{12}\,z' + q_{11}\,y + q_{12}\,z &= 0, \\ z'' + p_{21}\,y' + p_{22}\,z' + q_{21}\,y + q_{22}\,z &= 0, \end{aligned} \tag{1}$$

where

$$y' = \frac{dy}{dx}, \qquad y'' = \frac{d^2y}{dx^2}, \text{ etc.}$$

Its generators are the lines that join those points on the integral curves C_y and C_z which correspond to the same values of the independent variable x.

The differential equations of the integral curves C_y and C_z are‡

$$\begin{aligned} y^{(4)} + 4p_1\,y^{(3)} + 6p_2\,y'' + 4p_3\,y' + p_4\,y &= 0, \\ z^{(4)} + 4q_1\,z^{(3)} + 6q_2\,z'' + 4q_3\,z' + q_4\,z &= 0, \end{aligned} \tag{2}$$

where

* Certain theorems relating to the asymptotic curves on ruled surfaces with straight line directrices have been established by Cremona, Annali di matematiche, 1867–68, Halphen, Bulletin de la Societé mathématique de France, vol. 5 (1877), and Snyder, Bulletin of the American Mathematical Society, vol. 5 (1899).

† W, p. 126.

‡ W, p. 230.

$$p_1 = \frac{1}{4\Delta_1}(q_{12}\, l_{12} - p_{12}\, m_{12}), \qquad p_2 = \frac{1}{6\Delta_1}(s_{12}\, l_{12} - r_{12}\, m_{12}),$$

$$q_1 = \frac{1}{4\Delta_2}(q_{21}\, l_{21} - p_{21}\, m_{21}), \qquad q_2 = \frac{1}{6\Delta_2}(s_{21}\, l_{21} - r_{21}\, m_{21}),$$

$$p_3 = \frac{1}{4\Delta_1}[\, m_{12}(p_{12}\, r_{11} - p_{11}\, r_{12}) + l_{12}(p_{11}\, s_{12} - q_{12}\, r_{11}) - l_{11}\Delta_1\,],$$

(3)

$$p_4 = \frac{1}{\Delta_1}\,[\, m_{12}(p_{12}\, s_{11} - q_{11}\, r_{12}) + l_{12}(q_{11}\, s_{12} - q_{12}\, s_{11}) - m_{11}\Delta_1\,],$$

$$q_3 = \frac{1}{4\Delta_2}\,[\, m_{21}(p_{21}\, r_{22} - p_{22}\, r_{21}) + l_{21}(p_{22}\, s_{21} - q_{21}\, r_{22}) - l_{22}\Delta_2\,],$$

$$q_4 = \frac{1}{\Delta_2}\,[\, m_{21}(p_{21}\, s_{22} - q_{22}\, r_{21}) + l_{21}(q_{22}\, s_{21} - q_{21}\, s_{22}) - m_{22}\Delta_2\,];$$

and where we shall assume $\Delta_1 \neq 0$, $\Delta_2 \neq 0$; i. e., that neither C_y nor C_z is a plane curve.

The quantities Δ_1, Δ_2, l_{ij}, m_{ij}, s_{ij} are defined as follows:

$$\Delta_1 = p_{12}\, s_{12} - q_{12}\, r_{12}, \qquad \Delta_2 = p_{21}\, s_{21} - q_{21}\, r_{21},$$

$$r_{11} = p_{11}^2 + p_{12}\, p_{21} - p'_{11} - q_{11}, \qquad s_{11} = p_{11}\, q_{11} + p_{12}\, q_{21} - q'_{11},$$

$$r_{12} = p_{12}(p_{11} + p_{22}) - p'_{12} - q_{12}, \qquad s_{12} = p_{11}\, q_{12} + p_{12}\, q_{22} - q'_{12},$$

$$r_{21} = p_{21}(p_{11} + p_{22}) - p'_{21} - q_{21}, \qquad s_{21} = p_{21}\, q_{11} + p_{22}\, q_{21} - q'_{21},$$

$$r_{22} = p_{22}^2 + p_{12}\, p_{21} - p'_{22} - q_{22}, \qquad s_{22} = p_{21}\, q_{12} + p_{22}\, q_{22} - q'_{22},$$

$$l_{11} = -\, p_{11}\, r_{11} - p_{21}\, r_{12} + r'_{11} + s_{11}, \qquad m_{11} = -\, r_{11}\, q_{11} - r_{12}\, q_{21} + s'_{11},$$

$$l_{12} = -\, p_{12}\, r_{11} - p_{22}\, r_{12} + r'_{12} + s_{12}, \qquad m_{12} = -\, r_{11}\, q_{12} - r_{12}\, q_{22} + s'_{12},$$

$$l_{21} = -\, p_{11}\, r_{21} - p_{21}\, r_{22} + r'_{21} + s_{21}, \qquad m_{21} = -\, r_{21}\, q_{11} - r_{22}\, q_{21} + s'_{21},$$

$$l_{22} = -\, p_{12}\, r_{21} - p_{22}\, r_{22} + r'_{22} + s_{22}, \qquad m_{22} = -\, r_{21}\, q_{12} - r_{22}\, q_{22} + s'_{22}.$$

The fundamental seminvariants and invariants of the integral curve C_y are given by the following expressions*:

$$P_2 = p_2 - p'_1 - p_1^2, \qquad P_3 = p_3 - p''_1 - 3p_1\, p_2 + 2p_1^3,$$

(4)

$$P_4 = p_4 - 4p_1\, p_3 - 3p_2^2 + 12p_1^2\, p_2 - 6p_1^4 - p_1^{(3)},$$

$$\theta_3 = P_3 - \tfrac{3}{2}P'_2, \qquad \theta_4 = P_4 - 2P'_3 + \tfrac{6}{5}P''_2 - \tfrac{6}{25}P_2^2.$$

On replacing p, P, and θ by q, Q, and θ' respectively, we obtain the corresponding expressions for the integral curve C_z.

* W, p. 239.

Let the curves C_y and C_z be two of the *curved* asymptotic lines on the integrating ruled surface S of system (1). Then we have $p_{12} = p_{21} = 0$, and we may also assume that $p_{11} = p_{22} = 0$.*

It is easy to show and, moreover, it is well known, that an asymptotic curve on a ruled surface is never a plane curve unless it is a straight line. If we leave aside the case of a quadric surface, there will be at most two asymptotic curves of the second kind which are plane curves, and these will be straight lines. It follows that the curves C_y and C_z are distinct from these since, by hypothesis, they are not plane curves. The functions q_{12} and q_{21} must therefore be different from zero.

The tangents to the integral curves C_y and C_z will belong to linear complexes if, and only if, the invariants θ_3 and θ_3' vanish identically.† Let us assume, then, that system (1) has been so transformed that $p_{ij} = 0$ $(i = 1, 2; j = 1, 2)$; and let us express θ_3 and θ_3' in terms of the remaining coefficients of (1) by means of (2), (3), (4). We find the following values for θ_3 and θ_3';

$$(5) \quad \theta_3 = q_{11}' - q_{22}' - \frac{q_{12}'}{q_{12}}(q_{11} - q_{22}), \qquad \theta_3' = q_{11}' - q_{22}' - \frac{q_{21}'}{q_{21}}(q_{11} - q_{22}).$$

Thus, the conditions $\theta_3 = \theta_3' = 0$ reduce to

$$(6) \qquad q_{11}' - q_{22}' = \frac{q_{12}'}{q_{12}}(q_{11} - q_{22}), \qquad q_{11}' - q_{22}' = \frac{q_{21}'}{q_{21}}(q_{11} - q_{22}).$$

It may happen either that C_y and C_z are the *only* asymptotic curves on the surface S which belong to linear complexes, or else that there exists a third asymptotic curve on S which has the same property. In the latter case all the asymptotic curves on S have this property and, if the function $q_{11} - q_{22}$ should happen to vanish for the original pair of asymptotic curves, we may avoid this complication by choosing another pair of fundamental curves. We may therefore assume $q_{11} - q_{22}$ to be different from zero. If, however, C_y and C_z constitute the only pair of asymptotic curves on the surface which belong to linear complexes, they both belong to the same complex and we cannot avoid the vanishing of $q_{11} - q_{22}$ if C_y and C_z are to be asymptotic curves. In this case the invariant‡

$$\theta_9 = \begin{vmatrix} u_{11} - u_{22} & u_{12} & u_{21} \\ v_{11} - v_{22} & v_{12} & v_{21} \\ w_{11} - w_{22} & w_{21} & w_{21} \end{vmatrix}$$

also vanishes while its second order minors are not all equal to zero. The

* W, pp. 114, 142.

† W, p. 254.

‡ W, pp. 96, 167.

integral ruled surface itself belongs to a linear complex but has at most one straight line directrix.

Let us return to the case when $q_{11} - q_{22}$ is different from zero. The integration of (6) gives

$$q_{11} - q_{22} = aq_{12}, \qquad q_{11} - q_{22} = bq_{21},$$

where a and b are constants different from zero. In this case not only does the invariant θ_9 vanish, but so do all of its second order minors. Therefore the surface has a pair of straight line directrices coincident or distinct according as the invariant θ_4 of the surface S vanishes or not, i. e., according as the flecnodes of each generator are coincident or distinct.* The converse of this is also true.† Hence we have the theorem:

In order that all of the asymptotic curves of a ruled surface may belong to linear complexes the surface must have two straight line directrices, distinct or coincident.

If the invariant θ_4 of the ruled surface S does not vanish, the independent variable may be so chosen as to make θ_4 equal to any non-vanishing constant.‡ We may, therefore, assume the independent variable so chosen as to make the function $q_{11} - q_{22}$ equal to unity. The equations of the surfaces S will then assume the form

$$y'' + q_{11}\, y + az = 0, \qquad z'' + by + q_{22}\, z = 0, \tag{7}$$

where a and b are constants and where $q_{11} - q_{22} = 1$.

The differential equations (2) now become

$$y^{(4)} + (q_{11} + q_{22})\, y'' + 2q'_{11}\, y' + (q''_{11} + q_{11}\, q_{22} - ab)\, y = 0,$$

$$z^{(4)} + (q_{11} + q_{22})\, z'' + 2q'_{22}\, z' + (q''_{22} + q_{11}\, q_{22} - ab)\, z = 0.$$

From these it follows that *any curved asymptotic line on S is a projection of any other.*

In order to determine the ruled surfaces S all of whose asymptotic curves are twisted cubics, we must satisfy the further conditions $\theta_4 = \theta'_4 = 0$.§ We thus obtain the following equations for the functions q_{11} and q_{22}:

$$q_{11} - \tfrac{1}{2} = q_{22} + \tfrac{1}{2} = \varphi, \qquad \varphi'' + \tfrac{8}{5}\varphi^2 + c_1 = 0, \tag{8}$$

where

$$c_1 = -\tfrac{8}{5}(1 + 4ab).$$

If $\varphi' = 0$, then

$$q_{11} = k = \tfrac{1}{2} \pm \tfrac{8}{5}\sqrt{1 + 4ab}, \qquad q_{22} = k - 1,$$

so that q_{11} and q_{22} are constants.

* W, p. 169.
† W, p. 288.
‡ W, p. 117.
§ W, p. 242.

From these equations it follows that *the first derivative surface* is a projective transformation of the surface S itself, and the second derivative surface coincides with the surface S.*

The surface turns out to be the quartic

$$x_1^3 x_4 - x_2^3 x_3 = 0,\dagger$$

whose invariant θ_4 is equal to $16\,(4ab+1)$.

It therefore follows that the invariants θ_3 and θ_4 of the curve C_y may vanish, while the invariant θ_4 of the surface S is different from zero.

If $\varphi' \neq 0$, we can integrate the equation of condition (8) by means of elliptic functions. In fact, if we multiply both members of (8) by φ' and integrate, we find

$$\varphi'^2 = -\tfrac{4}{15}(4\varphi^3 - g_2\varphi - g_3),$$

where $g_2 = -\tfrac{1}{2}\tfrac{5}{2}C_1$ and g_3 is a constant of integration. Now it is always possible to construct a Weierstrass $\wp$-function with given invariants g_2, g_3 to satisfy this equation. We then find

$$q_{11} = \wp(u; g_2, g_3) + \tfrac{1}{2}, \qquad q_{22} = \wp(u; g_2, g_3) - \tfrac{1}{2}.$$

Equations (7) may be reduced to the form‡

$$(7a) \qquad \frac{d^2\eta}{du^2} + [A\wp + B]\,\eta = 0 \qquad \frac{d^2\zeta}{du^2} + [A\wp + B_1]\,\zeta = 0,$$

where A, B, B_1 are constants.

These equations are of the Picard type, and precisely those of such great importance in the analytical theory of heat. In fact, an integral of (7a) leads at once to an integral of the heat equation in elliptic coördinates. Since A is not of the form $n\,(n+1)$, the equations are not of the Lamé type.

If the directrices of the surface S coincide, i. e., if the invariant θ_4 vanishes, we can write equations (1) in the form

$$(9) \qquad \frac{d^2y}{dx^2} + q(x)\,y = 0, \qquad \frac{d^2z}{dx^2} + y + q(x)\,z = 0,$$

so that C_y is the straight line directrix and C_z is any curved asymptotic line on the surface. The condition (8) becomes

* W, pp. 187–188.

† To get this result, we transform (7) so that the fundamental curves coincide with the flecnode curves and then integrate the resulting equations.

‡ This reduction is effected by means of the following transformations:

$$u = \frac{2i}{\sqrt{15}}x, \qquad y = \alpha\eta + \beta\zeta, \qquad z = \gamma\eta + \delta\zeta,$$

where η and ζ are the linear factors of a certain quadratic covariant C (W, p. 124 et seq.).

$$\frac{d^2 q}{dx^2} + \tfrac{8}{5} q^2 = 0. \tag{8a}$$

If $q'' = 0$, q itself must vanish. In this case equations (9) show that the coördinates of any point on the surface S are given by the equations

$$x_1 = \alpha + \beta x^2, \qquad x_2 = \alpha x + \frac{\beta}{3} x^3, \qquad x_3 = -2\beta x, \qquad x_4 = -2\beta,$$

where α and β are arbitrary constants. Hence the equation of the surface is

$$x_3^3 + 3x_4 (x_1 x_3 - x_2 x_4) = 0,$$

which is that of a Cayley cubic scroll. If $q'' \neq 0$, equation (8a) gives on integration

$$\left(\frac{dq}{dx}\right)^2 = -\frac{4}{15}(4q^3 - g_3),$$

where g_3 is a constant. As before we construct the Weierstrass $\wp$-function satisfying this equation; so that equations (9) may be written

$$\frac{d^2 y}{du^2} + A\wp y = 0, \qquad \frac{d^2 z}{du^2} + A(\wp z + y) = 0, \tag{9a}$$

where A is a constant but not of the form $n(n+1)$.

Therefore *every ruled surface whose asymptotic curves are twisted cubics may be represented by a system of equations of the form* (7a) *or* (9a), *where* $\wp$ *is the Weierstrass elliptic function. If the coefficients of these equations reduce to constants, we obtain either a certain quartic or a Cayley cubic scroll.*

§ 2. Analytic criteria for non-ruled surfaces whose asymptotic curves belong to linear complexes

It is known that any non-developable surface S may be regarded as an integral surface of a non-involutory, completely integrable system of partial differential equations of the form*

$$y_{uu} + 2by_v + fy = 0, \qquad y_{vv} + 2a' y_u + gy = 0, \tag{10}$$

where the asymptotic curves are parametric, and where

$$y_u = \frac{\partial y}{\partial u}, \qquad y_{uu} = \frac{\partial^2 y}{\partial u^2}, \qquad \text{etc.}, \quad \cdots.$$

The integrability conditions of this system of equations are

$$\begin{gathered} a'_{uu} + g_u + 2ba'_v + 4a' b_v = 0, \qquad b_{vv} + f_v + 2a' b_u + 4ba'_u = 0, \\ g_{uu} - f_{vv} - 4fa'_u - 2a' f_u + 4gb_v + 2bg_v = 0. \end{gathered} \tag{11}$$

* M_1, p. 241.

The asymptotic curves $v = \text{const.}$ and $u = \text{const.}$ will be denoted by Γ' and Γ'' respectively.

The differential equations of Γ' and Γ'' are*

$$\begin{aligned} &y_{uuuu} + 4p_1' y_{uuu} + 6p_2' y_{uu} + 4p_3' y_u + p_4' y = 0, \\ &y_{vvvv} + 4p_1'' y_{vvv} + 6p_2'' y_{vv} + 4p_3'' y_v + p_4'' y = 0, \end{aligned} \tag{12}$$

where

$$p_1' = -\frac{b_u}{2b}, \qquad p_2' = \frac{1}{3}(f + b_v) - \frac{1}{6}\left(\frac{b_{uu}}{b} - 2\frac{b_u^2}{b^2}\right),$$

$$p_3' = \frac{1}{2}\left(f_u + 4a'b^2 - f\frac{b_u}{b}\right),$$

$$p_4' = f_{uu} - f^2 + 4b^2 g - 2bf_v + \frac{f}{b}(2bf + 2bb_v - b_{uu}) - 2\frac{b_u}{b^2}(bf_u - b_u f).$$

The fundamental seminvariants of Γ' have the values

$$\begin{aligned} P_2' &= p_2' - \frac{\partial p_1'}{\partial u} - p_1'^2, \\ P_3' &= p_3' - \frac{\partial^2 p_1'}{\partial u^2} - 3p_1' p_2' + 2p_1'^3, \\ P_4' &= p_4' - 4p_1' p_3' - 3p_2'^2 + 12p_1'^2 p_2' - 6p_1'^4 - \frac{\partial^3 p_1'}{\partial u^3}. \end{aligned} \tag{13}$$

Finally the fundamental invariants of Γ' are

$$\theta_3' = P_3' - \frac{3}{2}\frac{\partial P_2'}{\partial u}, \qquad \theta_4' = P_4' - 2\frac{\partial \theta_3'}{\partial u} - \frac{9}{15}\left(3\frac{\partial^2 P_2'}{\partial u^2} + \frac{2}{5}P_2'^2\right). \tag{14}$$

If we replace

$$a',\ b,\ f,\ g,\ p_i',\ P_i',\ \theta',\ u,\ v$$

by

$$b,\ a',\ g,\ f,\ p_i'',\ P_i'',\ \theta'',\ v,\ u$$

respectively, we obtain the corresponding expressions for Γ''.

In the subsequent applications of these equations we shall generally assume that $a' \not\equiv 0$, $b \not\equiv 0$, so that ruled surfaces are to be excluded unless the contrary is stated.

Let us consider the integral surfaces S for which the asymptotic curves Γ' and Γ'' belong to linear complexes. For these surfaces the invariants θ_3' and θ_3'' must vanish identically. By direct calculation from equations (12), (13), (14), these conditions are found to reduce to

* M_2, pp. 90, 93.

$$\Omega' \equiv 4a'b - \frac{\partial^2 \log a'}{\partial u\,\partial v} = 0,$$

(15)

$$\Omega'' \equiv 4a'b - \frac{\partial^2 \log b}{\partial u\,\partial v} = 0,$$

whence, by subtraction,

$$\frac{\partial^2 \log (a'/b)}{\partial u\,\partial v} = 0. \tag{16}$$

Hence, by integration,

$$b = \varphi(u)\,\theta(v)\,a'. \tag{17}$$

The transformation

$$y = \lambda(u, v)\,\bar{y}, \qquad \bar{u} = \alpha(u), \qquad \bar{v} = \beta(v) \tag{18}$$

leaves the parametric curves of equations (10) unchanged. Moreover the function $\lambda(u, v)$ may be chosen in such a way that the transformed system of equations shall again have the same (canonical) form as the system (10). The transformed coefficients $\bar{a}', \bar{b}, \bar{f}, \bar{g}$ will then be given by the equations

$$\bar{a}' = \frac{\alpha_u}{\beta_v^2}a', \qquad \bar{f} = \frac{1}{\alpha_u^2}(f - b\zeta - \tfrac{1}{2}\mu),$$

(19)

$$\bar{b} = \frac{\beta_v}{\alpha_u^2}b, \qquad g = \frac{1}{\beta_v^2}(g - a'\eta - \tfrac{1}{2}\nu),$$

where

$$\eta = \frac{\alpha_{uu}}{\alpha_u}, \qquad \zeta = \frac{\beta_{vv}}{\beta_v}, \qquad \mu = \eta_u - \tfrac{1}{2}\eta^2, \qquad \nu = \zeta_v - \zeta^2.*$$

In particular, the transformation

$$y = \frac{1}{\sqrt{\dfrac{d\bar{u}}{du}\cdot\dfrac{d\bar{v}}{dv}}}\cdot\bar{y}, \qquad \bar{u} = \int \sqrt[3]{\varphi(u)}\,du, \qquad \bar{v} = \int \frac{1}{\sqrt[3]{\theta(v)}}\,dv,$$

will replace equations (10) by a system of the same form in which the fundamental invariants a' and b are equal to each other. Let us assume that this transformation has been effected already, so that we have

$$a' = b.$$

The conditions

$$\Omega' = \Omega'' = 0$$

will then reduce to

$$\frac{\partial^2 \log b}{\partial u\,\partial v} = 4b^2. \tag{20}$$

If we put $b = e^{\phi/2}$, the above equation becomes

* M_1, pp. 249–250.

$$\frac{\partial^2 \phi}{\partial u \partial v} = 8e^{\phi}. \tag{20a}$$

But (20a) is the form known as Liouville's equation.* Its general integral is

$$e^{\phi} = \frac{1}{4} \frac{U'V'}{(U+V)^2},$$

where U and V are functions of the single variables u and v respectively, and where the accents indicate differentiation with respect to these variables. Hence the general integral of (20) is

$$a' = b = \frac{1}{2} \frac{\sqrt{U'V'}}{(U+V)}, \tag{21}$$

where we assume $U' \not\equiv 0$, $V' \not\equiv 0$ when we wish to exclude ruled surfaces. We shall refer to equations (10) as being in the *normal form*† when the invariants a' and b are given by (21). We therefore have the theorem:

Both families of asymptotic curves of a non-ruled surface belong to linear complexes, if and only if, the differential equations of the surface can be put in the normal form characterized by the values (21) *for the invariants a' and b.*‡

§ 3. Properties of the osculating ruled surfaces, and explicit determination of the differential equations of the surfaces of § 2

The differential equations of R_1 (one of the osculating ruled surfaces of the first kind) are§

$$\begin{aligned} y_{vv} + p_{11} y_v + p_{12} z_v + q_{11} y + q_{12} z = 0, \\ z_{vv} + p_{21} y_v + p_{22} z_v + q_{21} y + q_{22} z = 0, \end{aligned} \tag{22}$$

where $z = y_u$ and

$$\begin{aligned} p_{11} = p_{12} = p_{22} = 0, \quad q_{11} = g, \quad q_{12} = 2a', \\ p_{21} = -4a'b, \quad q_{21} = g_u - 2a'f, \quad q_{22} = g + 2a'_u. \end{aligned}$$

Since p_{12} is equal to zero, Γ'' is an asymptotic curve on R_1 as well as on the surface S.

The invariants of weights four and nine of R_1 are

$$\theta = (u_{11} - u_{22})^2 + 4u_{12}u_{21}, \tag{23}$$

and

$$\theta_9 = \begin{vmatrix} u_{11} - u_{22} & u_{12} & u_{21} \\ v_{11} - v_{22} & v_{12} & v_{21} \\ w_{11} - w_{22} & w_{12} & w_{21} \end{vmatrix},$$

* Darboux *Leçons sur la Théorie des Surfaces*, t. IV, pp. 419–424.

† The complete *normal form* will be given later.

‡ The invariants Ω' and Ω'' vanish for these surfaces only.

§ M_3, p. 81 et seq.

where

$$u_{11}-u_{22}=8a'_u,\qquad u_{12}=-8a',\qquad u_{21}=4a'_{uu}+8a'(f+b_v),$$

$$v_{11}-v_{22}=16(a'_{uv}-4a'^2b),\qquad v_{12}=-16a'_v,$$

$$v_{21}=8(a'_{uuv}-4a'a'_ub+2a'_vf+2a'f_v+2a'_vb_v+2a'b_{vv})$$

$$=8[a'_{uuv}+2a'_v(f+b_v)-4a'^2b_u-12a'ba'_u],$$

$$w_{11}-w_{22}=32(a'_{uvv}-4a'^2b_v-12a'ba'_v),\qquad w_{12}=-32a'_{vv},$$

$$w_{21}=16[a'_{uuvv}+2a'_{vv}(f+b_v)-4a'^2b_{uv}-12a'(a'_vb_u+a'_ub_v)$$

$$-16a'(ba'_{uv}-a'^2b^2)-20a'_ua'_vb],$$

the final forms of u_{21}, v_{21}, w_{21} resulting from an application of the integrability conditions (11).

Analogous equations may be obtained for R_2 (an osculating ruled surface of the second kind).

If we differentiate the conditions $\Omega'=0$, $\Omega''=0$, we obtain the following equations:

$$\begin{aligned}
&a'a'_{uuv}-a'_va'_{uu}-12a'^2ba'_u-4a'^3b_u=0,\\
&a'a'_{uvv}-a'_ua'_{vv}-12a'^2ba'_v-4a'^3b_v=0,\\
&bb_{uuv}-b_vb_{uu}-12a'b^2b_u-4b^3a'_u=0,\\
&bb_{uvv}-b_ub_{vv}-12a'b^2b_v-4b^3a'_v=0,\\
&a'a'_{uuvv}-a'_{uu}a'_{vv}-24a'ba'_ua'_v-12a'^2(a'_ub_v+a'_vb_u)\\
&\qquad-12a'^2ba'_{uv}-4a'^3b_{uv}=0,\\
&bb_{uuvv}-b_{uu}b_{vv}-24a'bb_ub_v-12b^2(a'_ub_v+a'_vb_u)\\
&\qquad-12a'b^2b_{uv}-4b^3a'_{uv}=0,
\end{aligned}\tag{24}$$

which enable us to reduce the elements of θ_9 to

$$u_{11}-u_{22}=8a'_u,\qquad u_{12}=-8a',\qquad u_{21}=4[a'_{uu}+2a'(f+b_v)],$$

$$v_{11}-v_{22}=16\frac{a'_ua'_v}{a'},\quad v_{12}=-16a_v,\quad v_{21}=\frac{8a'_v}{a'}[a'_{uu}+2a'(f+b_v)],$$

$$w_{11}-w_{22}=32\frac{a'_ua'_{vv}}{a'},\quad w_{12}=-32a'_{vv},\quad w_{21}=\frac{16a'_{vv}}{a'}[a'_{uu}+2a'(f+b_v)].$$

If we substitute these simplified expressions in θ_9, we find that θ_9 and all of its second order minors vanish. In the same way we find that θ'_9, the invariant of

weight nine for R_2, and all of its second order minors vanish. Consequently we have the theorem:

*If both families of asymptotic curves of a surface S belong to linear complexes, then both families of osculating ruled surfaces R_1 and R_2 have straight line directrices, distinct or coincident according as the corresponding invariant θ or θ' is different from zero or not.**

We shall need several other formulæ connected with the theory of the osculating ruled surfaces. In developing these formulæ we shall indicate another proof of the above theorem.

If P_η and P_ζ are the flecnodes on the generator $P_y P_z$ of R_1, we have†

$$y = 16a'(\eta + \zeta), \qquad z = (8a'_u + \sqrt{\theta})\,\eta + (8a'_u - \sqrt{\theta})\,\zeta.$$

When we make this transformation the equations of R_1, referred to its flecnode curves, are found to be

$$\begin{aligned} &\eta_{vv} + \pi_{11}\eta_v + \pi_{12}\zeta_v + \kappa_{11}\eta + \kappa_{12}\zeta = 0, \\ &\zeta_{vv} + \pi_{21}\eta_v + \pi_{22}\zeta_v + \kappa_{21}\eta + \kappa_{22}\zeta = 0, \end{aligned} \tag{25}$$

where

$$\pi_{11} = \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta}, \qquad \pi_{12} = \frac{a'_v}{a'} + \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta},$$

$$\pi_{21} = \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} - \frac{\theta_v}{2\theta}, \qquad \pi_{22} = \frac{a'_v}{a'} - \frac{C}{\sqrt{\theta}} + \frac{\theta_v}{2\theta},$$

$$C = 8a'_{uv} - 8\frac{a'_u a'_v}{a'} - 32a'^2 b.$$

But since we are dealing with the case when $\Omega' = 0$, we find $C = 0$. Consequently

$$\pi_{12} = \pi_{21} = \frac{a'_v}{a'} - \frac{\theta_v}{2\theta} = \frac{2^5}{a'\theta}\,[a'(a'a'_{uuv} - a'_{uu}a'_v - 12a'^2 ba'_u - 4a'^3 b) + a'_u(a'_u a'_v - a'a'_{uv} + 4a'^3 b)] = 0,$$

$$\pi_{11} = \pi_{22} = 2\frac{a'_v}{a'},$$

and therefore

$$\theta = \varphi(u)\,a'^2,$$

where φ is an arbitrary function of u. We find further

* Several months after this paper was presented to the American Mathematical Society, a memoir containing this theorem was published by Enrico Bompiani, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912).

† M_2, p. 83 et seq.

$$\kappa_{11} = \frac{1}{\Delta}\left[-4a'\theta - 32ga'\sqrt{\theta} - 32a'a_u'\sqrt{\theta} - \frac{8}{\sqrt{\theta}}\left(2a_{vv}'\theta + a'\theta_{vv} - \frac{a'\theta_v^2}{2\theta}\right)\right],$$

$$\kappa_{12} = \frac{-8}{\Delta\sqrt{\theta}}\left[2a_{vv}'\theta + a'\theta_{vv} - \frac{a'\theta_v}{2\theta}\right],^{*} \qquad \text{etc.},$$

which reduce to

$$\kappa_{11} = a_u' + g + \frac{\sqrt{\theta}}{8}, \qquad \kappa_{12} = 0, \qquad \kappa_{21} = 0, \qquad \kappa_{22} = a_u' + g - \frac{\sqrt{\theta}}{8}.$$

The equations of the osculating ruled surface R_1 may therefore be reduced to the form

$$\begin{aligned} &\eta_{vv} + \frac{\theta_v}{\theta}\eta_v + \left(a_u' + g + \frac{\sqrt{\theta}}{8}\right)\eta = 0, \\ &\zeta_{vv} + \frac{\theta_v}{\theta}\zeta_v + \left(a_u' + g - \frac{\sqrt{\theta}}{8}\right)\zeta = 0, \end{aligned} \tag{25a}$$

where $\theta / a'^2 = \varphi(u)$, a function of u alone.

The corresponding equations for R_2 may be obtained from the above by replacing θ, a', g by θ', b, f respectively, and interchanging u and v. Combining these results we have the following theorem:

If the invariants θ and θ' are different from zero, and if both families of asymptotic curves on S belong to linear complexes, the invariants θ / a'^2 and θ' / b^2† are functions of the single variables u and v respectively. The equations of the osculating ruled surfaces may be reduced to the form (25a), *which makes evident the already established theorem that the osculating ruled surfaces have straight line directrices.*

Suppose $\theta = 0$. Let P_ζ coincide with the double flecnode on $P_y P_z$. We make the transformation

$$y = \eta, \qquad z = \frac{a_u'}{2a'}\eta - \zeta,^{\ddagger}$$

so that the equations of R_1 referred to Γ'' and its flecnode curve are§

$$\begin{aligned} &\eta_{vv} + \frac{b_v^2 - 2bb_{vv}}{4b^2}\eta - 2a'\zeta = 0, \\ &\zeta_{vv} + \frac{b_v^2 - 2bb_{vv}}{4b^2}\zeta = 0, \end{aligned} \tag{26}$$

which are characteristic of ruled surfaces with coincident directrices.

* In order to obtain these results we must use relations (24), (11) and the relation

$$2a_{vv}'\theta + a'\theta_{vv} - a_v'\theta_v = 0$$

obtained by differentiating $\pi_{12} = \pi_{21} = 0$ with respect to v.

† $\theta' / b^2 = \psi(v)$, ψ being an arbitrary function of v.

‡ M₂, p. 86.

§ The reductions are effected by means of (24), (11), and the condition $\theta = 0$. The equations of R_2 referred to Γ' and its flecnode curve have a similar form when $\theta' = 0$.

We can now complete the determination of the *normal form* of the equations of S. We have

$$a' = b, \qquad \frac{\partial^2 \log b}{\partial u \partial v} = 4b^2,$$

and the integrability conditions (11), which are reducible to

$$b_{uu} + g_u + 6bb_v = 0, \qquad b_{vv} + f_v + 6bb_u = 0,$$

$$g_{uu} - f_{vv} - 4fb_u - 2bf_u + 4gb_v + 2bg_v = 0.$$

We have further (from the previous theorem and the definition of θ and θ'),

$$\theta = 64\,[b_u^2 - 2b\,(b_{uu} + 2b \cdot \overline{f + b_v})] = \varphi(u)\,b^2,$$

$$\theta' = 64\,[b_v^2 - 2b\,(b_{vv} + 2b \cdot \overline{g + b_u})] = \psi(v)\,b^2.$$

Whence

$$f = -b_v - \frac{1}{2}\frac{\partial^2 \log b}{\partial u^2} - \frac{1}{4}\frac{b_u^2}{b^2} + \varphi_1(u),$$

$$g = -b_u - \frac{1}{2}\frac{\partial^2 \log b}{\partial v^2} - \frac{1}{4}\frac{b_v^2}{b^2} + \varphi_2(v),$$

where

$$\varphi_1(u) = -\frac{\varphi(u)}{256}, \qquad \varphi_2(v) = -\frac{\psi(v)}{256}.$$

Substituting these values of f and g in the integrability conditions, we find that the first two are satisfied identically and the third becomes

$$-4b_u\,\varphi_1(u) + 4b_v\,\varphi_2(v) - 2b\,\varphi_1'(u) + 2b\,\varphi_2'(v) = 0.$$

Now b is given by equation (21), so that

$$\frac{b_u}{b} = \frac{1}{2}\frac{U''}{U'} - \frac{U'}{U+V}, \qquad \frac{b_v}{b} = \frac{1}{2}\frac{V''}{V'} - \frac{V'}{U+V}.$$

We must therefore have

$$2\frac{\varphi_1(u)\,U' - \varphi_2(v)\,V'}{U+V} + \varphi_2(v)\frac{V''}{V'} - \varphi_1(u)\frac{U''}{U'} = \varphi_1'(u) - \varphi_2'(v).$$

Thus the left hand member of this equation reduces to a function of u minus a function of v. Now U' and V' can not vanish, for then b also would vanish. Hence U and V are not constants, and we must have

$$\frac{\varphi_1(u)\,U' - \varphi_2(v)\,V'}{U+V} = H_1(u) - H_2(v)$$

or

$$(\alpha) \quad \varphi_1(u)\,U' - \varphi_2(v)\,V' = UH_1(u) - VH_2(v) + VH_1(u) - UH_2(v),$$

where φ_i, H_i are functions to be determined. These must be such that $VH_1(u) - UH_2(v)$ is of the form $G_1(u) - G_2(v)$, that is we must have

$$\frac{\partial^2}{\partial u \partial v}(VH_1(u) - UH_2(v)) = 0,$$

whence

$$\frac{1}{U'}\frac{dH_1}{du} = \frac{1}{V'}\frac{dH_2}{dv}.$$

Since the left hand member is a function of u alone and the right hand member is a function of v alone, their equality implies

$$\frac{1}{U'}\frac{dH_1}{du} = k_0, \quad \frac{1}{V'}\frac{dH_2}{dv} = k_0,$$

where k_0 is a constant. Therefore

$$H_1(u) = k_0 U + k_1, \qquad H_2(v) = k_0 V + k_2,$$

where k_1 and k_2 are additional constants. If these values for H_1 and H_2 be substituted in (α), we find

$$\varphi_1(u) = \frac{aU^2 + bU + c}{U'}, \qquad \varphi_2(v) = \frac{aV^2 - bV + c}{V'},$$

where $a = k_0$, $b = k_1 - k_2$, and c is a further constant. If we substitute these values of $\varphi_1(u)$ and $\varphi_2(v)$ in the expressions for f and g, we shall find

(27)
$$f = -\frac{1}{2}\frac{\sqrt{U'V'}}{(U+V)}\left(\frac{V''}{2V'} - \frac{V'}{(U+V)}\right) - \frac{1}{4}\left(\frac{U''}{U'} - \frac{3}{4}\left(\frac{U''}{U'}\right)^2\right) + \frac{3}{4}\left(\frac{U''}{U+V} - \frac{U'^2}{(U+V)^2}\right) + \frac{aU^2 + bU + c}{U'},$$

$$g = -\frac{1}{2}\frac{\sqrt{U'V'}}{(U+V)}\left(\frac{U''}{2U'} - \frac{U'}{(U+V)}\right) - \frac{1}{4}\left(\frac{V''}{V'} - \frac{3}{4}\left(\frac{V''}{V'}\right)^2\right) + \frac{3}{4}\left(\frac{V''}{U+V} - \frac{V'^2}{(U+V)^2}\right) + \frac{aV^2 - bV + c}{V'}.$$

If the osculating ruled surfaces are characterized by equations of the form (26), it is readily seen that $a = b = c = 0$, in the above equations. A discussion of some of the geometrical configurations corresponding to this case will be given below. We may recapitulate these results in the following theorem:

Every non-ruled surface all of whose asymptotic curves belong to linear complexes is defined by a system of equations of the form (10), *where the fundamental invariants* a', b *and seminvariants* f, g *are given explicitly as functions of* u *and* v

by equations (21) *and* (27), *and conversely. Since all the invariants of* S *are functions of* a', b, f, g *and their derivatives, they can be determined as explicit functions of* u *and* v.

§ 4. Properties of the linear complexes of the asymptotic curves, and a geometric construction for the surfaces of the problem

The directrices δ_1 and δ_2 of the surface R_1 are found to have the equations*

$$\begin{aligned} 2^4 a' x_1 + A x_2 + 2^5 a'^2 b x_4 &= 0, \\ 2^4 a' x_3 + A x_4 &= 0, \end{aligned} \tag{28}$$

and

$$\begin{aligned} 2^4 a' x_1 + B x_2 + 2^5 a'^2 b x_4 &= 0, \\ 2^4 a' x_3 + B x_4 &= 0, \end{aligned}$$

where

$$A = 8a'_u - \sqrt{\theta}, \qquad B = 8a'_u + \sqrt{\theta}.$$

The equations of the directrices δ'_1 and δ'_2 of R_2 are found to be

$$\begin{aligned} 2^4 b x_1 + A_1 x_3 + 2^5 a' b^2 x_4 &= 0, \\ 2^4 b x_2 + A_1 x_4 &= 0, \end{aligned} \tag{28a}$$

and

$$\begin{aligned} 2^4 b x_1 + B_1 x_3 + 2^5 a' b^2 x_4 &= 0, \\ 2^4 b x_2 + B_1 x_4 &= 0, \end{aligned}$$

where

$$A_1 = 8b_v - \sqrt{\theta'}, \qquad B_1 = 8b_v + \sqrt{\theta'}.$$

These equations are of fundamental importance in the succeeding investigation.

It is clear that, when Ω' and Ω'' vanish, the osculating linear complexes of the curves Γ' and Γ'' coincide with the linear complexes which contain all of the tangents to these curves. Hence, referred to the moving tetraedron $P_y P_z P_\rho P_\sigma$, the linear complexes determined by Γ' and Γ'' are given by the equations†

$$\begin{aligned} &(a) \quad C' \equiv -b_v \omega_{34} - b\omega_{14} + b\omega_{23} = 0, \\ &(b) \quad C'' \equiv -a'_u \omega_{42} + a' \omega_{14} + a' \omega_{23} = 0. \end{aligned} \tag{29}$$

The invariants of C' and C'' are $-b^2$ and a'^2 respectively.

We shall now determine the linear complexes to which belong the asymptotic curves through the neighbouring points $P_{u+du,\,v}$ and $P_{u,\,v+dv}$. Their

* $P_y P_z P_\rho P_\sigma$ is the tetrahedron of reference. In this system the coördinates of the point $P \equiv \alpha y + \beta z + \gamma\rho + \delta\sigma$ are, by definition, $(\alpha, \beta, \gamma, \delta)$.

† M_2, pp. 92–94.

equations will, of course, have the same form as (29), the tetrahedra of reference being those associated with the points considered. Now when u receives an increment du, v remaining constant, the semi-covariants y, z, ρ, σ change. Let $\bar{y}$, $\bar{z}$, $\bar{\rho}$, $\bar{\sigma}$ be their new values.* We shall find

$$
\begin{aligned}
\bar{y} &= y + z\,du + \cdots, \\
\bar{z} &= z - (fy + 2b\rho)\,du + \cdots, \\
\bar{\rho} &= \rho + \sigma du + \cdots, \\
\bar{\sigma} &= \sigma + [(2bg - f_v)\,y + 4a'\,bz - (2b_v + f)\,\rho]\,du + \cdots.
\end{aligned} \tag{30}
$$

Let the coördinates of any point Q referred to T and $\bar{T}$ be (x_1, x_2, x_3, x_4) and $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ respectively. Then the point Q is given by each of the expressions

$$x_1 y + x_2 z + x_3 \rho + x_4 \sigma, \qquad \bar{x}_1 \bar{y} + \bar{x}_2 \bar{z} + \bar{x}_3 \bar{\rho} + \bar{x}_4 \bar{\sigma},$$

which must therefore be identical except for a factor. Making use of this fact and equations (30), we find (on neglecting du^2 and higher powers of du)

$$
\begin{aligned}
\omega x_1 &= \bar{x}_1 + (-f\bar{x}_2 + \alpha\bar{x}_4)\,du, \\
\omega x_2 &= \bar{x}_1\,du + \bar{x}_2 + 4a'\,b\bar{x}_4\,du, \\
\omega x_3 &= -2b\bar{x}_2\,du + \bar{x}_3 - \beta\bar{x}_4\,du, \\
\omega x_4 &= \bar{x}_3\,du + \bar{x}_4.
\end{aligned} \tag{31}
$$

Whence

$$
\begin{aligned}
\omega'\,\bar{x}_1 &= x_1 + (fx_2 - \alpha x_4)\,du, \\
\omega'\,\bar{x}_2 &= -x_1\,du + x_2 - 4a'\,bx_4\,du, \\
\omega'\,\bar{x}_3 &= 2bx_2\,du + x_3 + \beta x_4\,du, \\
\omega'\,\bar{x}_4 &= -x_3\,du + x_4,
\end{aligned}
$$

where ω and ω' are factors of proportionality and where

$$\alpha = 2bg - f_v, \qquad \beta = 2b_v + f.$$

If ω_{ij} and $\bar{\omega}_{ij}$ are the coördinates of a line referred to T and $\bar{T}$ respectively, we shall have

$$
\begin{aligned}
\bar{\omega}_{14} &= \omega_{14} + (-\omega_{13} - f\omega_{42})\,du, \\
\bar{\omega}_{23} &= \omega_{23} + (-\omega_{14} + 4a'\,b\omega_{34} - \beta\omega_{42})\,du, \\
\bar{\omega}_{42} &= \omega_{42} + (\omega_{14} + \omega_{23})\,du.
\end{aligned}
$$

* The tetrahedron $T \equiv P_y P_z P_\rho P_\sigma$ is displaced to $\bar{T} \equiv P_{\bar{y}} P_{\bar{z}} P_{\bar{\rho}} P_{\bar{\sigma}}$.

The linear complex C''_{u+du}, containing the asymptotic curve Γ''_{u+du}, is given by the equation

$$-\bar{a}'_u\,\bar{\omega}_{42}+\bar{a}'\,(\bar{\omega}_{14}+\bar{\omega}_{23})=0, \tag{32}$$

referred to $\bar{T}$, where

$$\bar{a}'=a'+a'_u\,du+\cdots,$$

$$\bar{a}'_u=a'_u+a'_{uu}\,du+\cdots.$$

Therefore the equation of C''_{u+du}, referred to T, is

$$\begin{aligned}&(-a'_u\,\omega_{42}+a'\,\omega_{14}+a'\,\omega_{23})\\ &\qquad+(-2a'\,\omega_{13}+4a'^2\,b\omega_{34}+\delta\omega_{42})\,du=0,\end{aligned} \tag{32a}$$

where

$$\delta=-a'_{uu}-2a\,(f+b_v)=\frac{\theta-2^6\,a'^2_u}{2^7\,a'}.$$

We shall need the invariant $\bar{I}''$ of the complex

$$-2a'\,\omega_{13}+4a'^2\,b\omega_{34}+\delta\omega_{42}=0,$$

and the mutual invariant $(I'',\bar{I}'')$ of this complex and C''_u. We find

$$\bar{I}''=-2a'\,\delta,\qquad (I'',\bar{I}'')=2a'\,a'_u.$$

The complexes C''_u an dC''_{u+du} determine the pencil

$$\lambda\,(-a'_u\,\omega_{42}+a'\omega_{14}+a'\,\omega_{23})+\mu\,(-2a'\,\omega_{13}+4a'^2\,b\omega_{34}+\delta\omega_{42})=0. \tag{33}$$

The special complexes of this pencil are given by the values of λ and μ which satisfy the equation

$$\lambda^2\,I''+\lambda\mu\,(I'',\bar{I}'')+\mu^2\,\bar{I}''=0.$$

Hence

$$\frac{\lambda}{\mu}=\frac{-a_u\pm\frac{\sqrt{\theta}}{2^3}}{a'}.$$

Introducing these values of λ/μ into equation (33), we find the line coördinates of the directrices of the special complexes of the pencil to be

$$\begin{aligned}&\omega_{12}=4a'^3\,b,\qquad \omega_{13}=a'^2_u\mp\frac{a'_u\sqrt{\theta}}{2^3}+a'\,\delta,\qquad \omega_{14}=a'\left(-a'_u\pm\frac{\sqrt{\theta}}{2^3}\right),\\ &\omega_{23}=a'\left(-a'_u\pm\frac{\sqrt{\theta}}{2^3}\right),\qquad \omega_{34}=0,\qquad \omega_{42}=-2a'^2.\end{aligned} \tag{34}$$

Their point equations are

$$\begin{aligned}&2^4\,a'\,x_1+(8a'_u\mp\sqrt{\theta})\,x_2+2^5\,a'^2\,bx_4=0,\\ &2^4\,a'\,x_3+(8a'_u\mp\sqrt{\theta})\,x_4=0.\end{aligned} \tag{34a}$$

If we substitute the coördinates (34) in equation (29a), we find that the directrices of the congruence determined by the complexes C''_u and C''_{u+du} (i. e., according to equation (28) the directrices of R_1) belong to the complex C'_v. Similarly we prove that the directrices of R_2 belong to C''_u. Hence the theorem:

The directrices of the osculating ruled surfaces R_1 are contained in the complexes C'_v determined by Γ', and the directrices of R_2 are contained in the complexes C''_u determined by Γ''.

It is of interest to obtain this result in another way. Let H be the common osculating hyperboloid of R_1 and R_2. Its equation is

$$x_1 x_4 - x_2 x_3 + 2a' b x_4^2 = 0.^* \tag{35}$$

The lines (34a) and the corresponding lines of R_2 are easily seen to be on H. The complex C''_u contains three generators of the first kind on H and therefore all the generators of this set. The complex C'_v contains three and therefore all the generators of the second set on H. The directrices (δ_1, δ_2) and (δ'_1, δ'_2) are among these generators, as we have just seen. This proves the theorem.

Let v remain fixed and let u vary. Then R_2 remains fixed as P_y describes the asymptotic curve Γ'_v on S. Hence the directrices (δ'_1, δ'_2) of R_2 belong to all of the complexes C''_u. But R_2 is any surface of the one parameter family of osculating ruled surfaces of the second kind. Hence all the directrices of R_2 are generators of a ruled surface S_2 contained in all of the complexes C''_u of S. Similarly the directrices of R_1 are generators of a ruled surface S_1 contained in all of the complexes C'_v of S. We shall now prove that S_1 is identical with S_2 and therefore a quadric, the two sets of directrices being complementary reguli.

Since the four directrices (δ_1, δ_2) and (δ'_1, δ'_2) of the osculating ruled surfaces R_1 and R_2 determined by Γ' and Γ'' are situated upon the osculating hyperboloid H of the point P_y, and since this is true for every point of Γ' when v is fixed, and for every point of Γ'' when u is fixed, it follows that all the directrices of R_1 intersect all the directrices of R_2. Therefore the generators of S_1 and S_2 constitute complementary reguli on the same quadric surface. We shall call this surface the *directrix quadric* and denote it, for brevity, by the letter Q.

If the integrating surface S is not ruled, all the complexes C' can not be those of a pencil of complexes. For, if all the complexes C' were contained in a pencil, the congruence determined by C'_v and C'_{v+dv} would have the same pair of directrices (δ'_1, δ'_2) for every v. Therefore all the osculating ruled surfaces

* M_2, p. 82.

of one family would belong to the same linear congruence, viz., the one having (δ_1', δ_2') as directrices. But this implies that S itself is a ruled surface of the congruence. Similarly we show that the complexes C'' are not all contained in a pencil.

Therefore *if all the linear complexes of the asymptotic curves of either family belong to a pencil, the surface must be ruled.*

If Γ' or Γ'' belong to a one parameter family of linear complexes not contained in a net, then these complexes could not intersect in more than two lines.* Therefore, there could not be more than one pair of directrices for the osculating ruled surfaces of the other family. But by the above theorem the surface S would, in this case, be ruled. We therefore have the theorem:

If both families of asymptotic curves of a non-ruled surface S belong to linear complexes, these complexes must form two one-parameter families belonging to two involutory nets.†

This theorem, together with one which we shall now prove, leads to an important result, namely, *a geometric construction for the surfaces having the property in question.*

The osculating hyperboloid $H_{u+du,\,v}$ of the surfaces R_1 and R_2 determined by the asymptotic curves through the point $P_{u+du,\,v}$ has the equation

$$\bar{x}_1 \bar{x}_4 - \bar{x}_2 \bar{x}_3 + 2\bar{a}' \bar{b} \bar{x}_4^2 = 0,$$

referred to $\bar{T}$. This becomes, when referred to T,

$$(x_1 x_4 - x_2 x_3 + 2a' b x_4^2) + (-2b x_2^2 - 2b_v x_2 x_4 + \gamma x_4^2)\, du = 0,$$

where

$$\gamma = 2\,[(a'\,b)_u - bg].$$

Now $H_{u,\,v}$ and $H_{u+du,\,v}$ intersect along two generators of R_2 consecutive to $P_y P_\rho$, and two generators of the second set upon H in the planes

$$2^4 b x_2 - (8b_v \pm \sqrt{\theta'})\, x_4 = 0\ddagger. \tag{36}$$

The tangent plane to $H_{u,\,v}$ at the point $(y_1, 0, y_3, 0)$ is

$$x_2 y_3 - x_4 y_1 = 0. \tag{36a}$$

Therefore the generators of intersection of $H_{u,\,v}$ and $H_{u+du,\,v}$, apart from those consecutive to $P_y P_\rho$, pass through the points

$$\omega\eta = (8b_v - \sqrt{\theta'})\, y - 16b\rho, \qquad \omega\zeta = -(8b_v + \sqrt{\theta'})\, y + 16b\rho. \tag{36b}$$

These two generators are the flecnode tangents to R_2 at the flecnodes on $P_y P_\rho$.

* The complexes of a hypernet have, at most, two lines in common.

† The mutual invariant (I'', I') vanishes and therefore the complexes are in involution.

‡ This follows from the definition of θ' and the second integrability condition.

Let us now assume that the asymptotic curves Γ'' belong to a one parameter family of linear complexes contained in a net, but not contained in a pencil. The complexes of this net have in common a regulus ρ'' of a quadric Q, as we have seen. The directrices $(\delta_1(u, v), \delta_2(u, v))$ of R_1 are situated upon the complementary regulus ρ' of Q, and also upon the osculating hyperboloid $H_{u,v}$. Hence Q and $H_{u,v}$ intersect in another pair of generators $(g_1(u), g_2(u))$ of Q and $H_{u,v}$ which belong to the regulus ρ''. Again, the directrices $(\delta_1(u+du, v), \delta_2(u+du, v))$ of $R_{1_{u+du,v}}$ are on Q and belong to ρ'; hence Q and $H_{u+du,v}$ intersect in another pair of generators $(g_1(u+du), g_2(u+du))$ which belong to ρ''. Now the generators, not consecutive to $P_y P_\rho$, common to $H_{u,v}$ and $H_{u+du,v}$ are given by equations (36), (36*a*), (36*b*). These being on $H_{u,v}$ and $H_{u+du,v}$ must intersect the four generators $(\delta_1(u, v), \delta_2(u, v))$ and $(\delta_1(u+du, v), \delta_2(u+du, v))$ of Q and are, therefore, generators of the regulus ρ''. If we call these lines (l_1, l_2), it follows that the pairs $(g_1(u), g_2(u))$ and $(g_1(u+du), g_2(u+du))$ both coincide with the pair (l_1, l_2). For otherwise the total intersection of Q and $H_{u,v}$ would be of higher order than the fourth, i. e., Q and $H_{u,v}$ would coincide and S would be itself a quadric surface. In precisely the same way we see that $H_{u+du,v}$, etc., $\cdots$, intersect in two generators belonging to ρ'' of Q. These must, therefore, be (l_1, l_2). Proceeding in this manner we find that all of the osculating hyperboloids of $R_2(v_0)$ (say) have in common a fixed pair of lines of the regulus ρ'' of Q, viz., the pair (l_1, l_2), and therefore R_2 has a pair of straight line directrices* $(\delta_1(u, v_0), \delta_2(u, v_0))$. Hence all of its asymptotic curves belong to linear complexes. In particular Γ' belongs to a linear complex, and this is true for every value of v. We therefore have the theorem:

If the asymptotic curves of one family of a non-ruled surface belong to a one-parameter family of linear complexes contained in a net, then the asymptotic curves of the second family must also belong to a one-parameter family of linear complexes contained in an involutory net.

To construct the surfaces having the property in question we may now proceed as follows: Select a one-parameter family of pairs of lines $(\delta_1(u), \delta_2(u))$ on Q. Each of these pairs of lines determines a pencil of linear complexes, $(\delta_1(u), \delta_2(u))$ being the axes of the special complexes of this pencil. In each of these pencils pick out (according to some arbitrary law) one complex $C(u)$. We thus obtain a one-parameter family of complexes contained in a net. Every line which intersects $(\delta_1(u), \delta_2(u))$ belongs to $C(u)$. Let us begin with $u = u_0$. Let $R_1(u_0)$ be an arbitrary ruled surface having $(\delta_1(u_0), \delta_2(u_0))$ as directrices. $R_1(u_0)$ will then belong to $C(u_0)$, and one of its asymptotic curves $\Gamma(u_0)$ intersecting every generator twice will likewise belong to

* $(\delta_1(u, v_0), \delta_2(u, v_0))$ are the lines (l_1, l_2).

the complex $C(u_0)$. Let $g^{(0)}_{u_0}$, $g^{(1)}_{u_0}$, $g^{(2)}_{u_0}$, etc., be the generators of $R_1(u_0)$ corresponding to $v = v_0$, $v_0 + dv$, $v_0 + 2dv$, etc., and intersecting $\Gamma(u_0)$ in the points A_0, A_1, A_2, etc.

Now let $(\delta_1(u_0 + du), \delta_2(u_0 + du))$ be the pair of lines of Q which correspond to $u = u_0 + du$, and $C(u_0 + du)$ the corresponding linear complex. Let P_0 be a point of $g^{(0)}_{u_0}$ ultimately to be allowed to approach the corresponding point A_0 of $\Gamma(u_0)$ as its limit. The line $g^{(0)}_{u_0+du}$ through P_0 which intersects $(\delta_1(u_0 + du), \delta_2(u_0 + du))$ belongs to the complex $C(u_0 + du)$. Hence the null-plane of P_0 in the complex $C(u_0 + du)$ contains $g^{(0)}_{u_0+du}$. Let its intersection with $g^{(1)}_{u_0}$ be called P_1. Then P_0P_1 also belongs to $C(u_0 + du)$. Denote by $g^{(1)}_{u_0+du}$ the line through P_1 which intersects $(\delta_1(u_0 + du), \delta_2(u_0 + du))$, and by P_2 the point where $g^{(2)}_{u_0}$ is cut by the null-plane of P_1 in the null-system of $C(u_0 + du)$. Then $g^{(1)}_{u_0+du}$ and P_1P_2 are lines of the complex $C(u_0 + du)$. As we continue this process and proceed to the limit (i. e., $dv \doteq 0$) we obtain a curve on $R_1(u_0 + du)$ whose tangents obviously belong to the complex $C(u_0 + du)$. Moreover, the null-planes of the points of this curve in the null-system of the complex $C(u_0 + du)$ are clearly tangent to the surface $R_1(u_0 + du)$. But for any curve of a linear complex the osculating plane at any point of the curve coincides with its null-plane. Therefore the curve $R_1R_2R_3\cdots$ is an asymptotic curve of $R_1(u_0 + du)$.

By starting anew with the lines $(\delta_1(u_0 + 2du), \delta_2(u_0 + 2du))$ and the surface $R_1(u_0 + du)$, etc., we can construct a new ruled surface $R_1(u_0 + 2du)$ with an asymptotic curve on it belonging to the complex $C(u_0 + 2du)$, etc. If now we allow du to approach zero, we (finally) obtain a surface S as the envelope of the ruled surfaces constructed in this way; and the asymptotic curves of one family on S clearly belong to the given one parameter family of linear complexes. But by a previous theorem it follows that the asymptotic curves of the second family on S will also belong to linear complexes.

If the asymptotic curves of a single family belong to linear complexes (i. e., if $\Omega' = 0$, $\Omega'' \neq 0$), we select a one parameter family of pairs of lines $(\delta_1(u), \delta_2(u))$ on a directrix ruled surface of the third or higher order so as to preclude the possibility of the second family of asymptotic curves belonging to linear complexes. The remaining steps in this construction are precisely the same as in the case just considered and need not be repeated. Our construction shows that this directrix ruled surface may be chosen arbitrarily.

§ 5. The lines common to the complexes C'_v, C''_u, C'_{v+dv}, C''_{u+du}*

The complexes C'_v, C''_u, C'_{v+du}, C''_{u+du} have in common the same lines as the complexes

* Cf. M_2, p. 97.

$$
\begin{aligned}
-a'_u\,\omega_{42} + a'\,\omega_{14} + a'\,\omega_{23} &= 0,\\
-2a'\,\omega_{13} + 4a'^2\,b\omega_{34} + \delta\omega_{42} &= 0,\\
-b_v\,\omega_{34} - b\omega_{14} + b\omega_{23} &= 0,\\
+2b\omega_{12} + 4a'\,b^2\,\omega_{42} + \delta'\,\omega_{34} &= 0,
\end{aligned}
$$

where

$$\delta = \frac{\theta - 2^6\,a'^2_u}{2^7\,a'}, \qquad \delta' = \frac{\theta' - 2^6\,b_v^2}{2^7\,b}.$$

All of the lines common to these complexes intersect the lines

$$2a'\,bx_1 + a'_u\,bx_2 + a'\,b_v\,x_3 = 0, \qquad x_4 = 0;$$

$$2bx_2 + b_v\,x_4 = 0, \qquad 2a'\,x_3 + a'_u\,x_4 = 0,$$

which are the directrices d and d' of the first and second kinds of the point $P_{y(u,\,v)}$.* If

$$p = -a'_u\,y + 2a'\,z, \qquad q = -b_v\,y + 2b\rho,$$

then $\lambda p + \mu q$ is an arbitrary point on the directrix of the first kind. Similarly, if we write

$$r = -a'\,b_v z - ba'_u\,\rho + 2a'\,b\sigma,$$

then $\lambda'\,y + \mu'\,r$ is an arbitrary point on the directrix of the second kind. The coördinates of the line joining these points are

$$
\begin{aligned}
\omega_{12} &= -2a'\,\lambda\lambda' + a'\,a'_u\,b_v\,\lambda\mu' + a'\,b_v^2\,\mu\mu',\\
\omega_{13} &= -2b\mu\lambda' + ba'^2_u\,\lambda\mu' + bb_v\,a'_u\,\mu\mu',\\
\omega_{14} &= -2a'\,ba'_u\,\lambda\mu' - 2a'\,bb_v\,\mu\mu',\\
\omega_{23} &= -2a'\,ba'_u\,\lambda\mu' + 2a'\,bb_v\,\mu\mu',\\
\omega_{34} &= 4a'\,b^2\,\mu\mu', \qquad \omega_{42} = -4a'^2\,b\lambda\mu'.
\end{aligned}
$$

In order that this line may belong to the complexes C'_{v+dv} and C''_{u+du}, λ, μ, λ', μ' must satisfy the equations

$$
\begin{aligned}
\theta\lambda\mu' - 2^7\,\lambda'\,\mu + 2^6\,(a'_u\,b_v - 2^3\,a'^2\,b^2)\,\mu\mu' &= 0,\\
2^7\,\lambda\lambda' - 2^6\,(a'_u\,b_v - 2^3\,a'^2\,b^2)\,\lambda\mu' - \theta'\,\mu\mu' &= 0.
\end{aligned}
$$

Hence

$$\frac{\lambda}{\mu} = \pm\sqrt{\frac{\theta'}{\theta}}, \qquad \frac{\lambda'}{\mu'} = \frac{(a'_u\,b_v - 2^3\,a'^2\,b^2) \pm \frac{\sqrt{\theta\theta'}}{2^6}}{2}.$$

* M_2, pp. 95, 96.

Thus we have two lines* common to these four complexes, and they are obviously the diagonals of the skew quadrilateral $(\delta_1, \delta_2, \delta_1', \delta_2')$. Since $\lambda/\mu = \pm\sqrt{\theta'/\theta}$, l_1 and l_2 intersect the directrix d in two points which are harmonic conjugates with respect to p and q. Again d' intersects the osculating hyperboloid H in the points $P(1, 0, 0, 0)$ and $Q(x_1, x_2, x_3, x_4)$, where

$$x_1 = (a_u' b_v - 2^3 a'^2 b^2), \qquad x_2 = -2a' b_v, \qquad x_3 = -2a_u' b, \qquad x_4 = 4a' b.$$

Hence l_1 and l_2 intersect d' in two points which are harmonic conjugates with respect to P and Q. We therefore have the theorem:

If the asymptotic curves of a non-ruled surface belong to linear complexes, and if the directrices of R_1 and R_2 are distinct, the diagonals of the net-work of skew quadrilaterals formed by the directrices of R_1 and R_2 on the directrix quadric are the lines common to C_v', C_u'', C_{v+dv}', C_{u+du}''. These lines intersect the directrices of the first kind in points which are harmonic conjugates with respect to the intersections of the latter with the asymptotic tangents. They intersect the directrices of the second kind in points which are harmonic conjugates with respect to the intersections of the latter with the osculating hyperboloid.

§ 6. Applications to the theory of ruled surfaces

If the integrating surface S is a ruled surface not a quadric, we may assume a' equal to zero, and b different from zero. Then from equations (11) and

$$\Omega' \equiv b_{uv} - \frac{b_u b_v}{b},$$

it follows that we can find by quadratures a transformation of the type (18) which replaces equations (10) by the following system:

$$y_{uu} + 2y_v + fy = 0, \qquad y_{vv} - ay = 0, \tag{37}$$

where a is a constant and f is a function of u alone, since, by equations (11), $g_u = g_v = f_v = 0$. Therefore *any non-developable ruled surface (not a quadric) whose asymptotic curves belong to linear complexes may be represented by a system of equations of the form* (37), *and conversely, every such system of equations defines a projectively equivalent family of non-developable surfaces having the property in question.*

From (37) we find that

$$\theta' = 2^8 a, \qquad \theta_4 = \frac{2}{5} f_{uu} + \frac{16}{25} f^2 - 4a.\dagger$$

* We shall refer to these as l_1 and l_2.

† θ' denotes the invariant of weight four for the surface S and θ_4 the invariant of weight four for Γ''.

If we impose the additional condition that the asymptotic curves be twisted cubics, we find precisely the same equations as we obtained from the point of view of ordinary differential equations. Corresponding to the first case under (9) we find

$$\theta' = a = f = 0.$$

Equations (37) now become

$$y_{uu} + 2y_v = 0, \qquad y_{vv} = 0.$$

On integrating the second equation we find

$$y = U_1 v + U_2,$$

where U_1 and U_2 are functions of u alone. Substituting this expression for y in the first equation and integrating the resulting equations

$$\frac{d^2 U_1}{du^2} = 0, \qquad \frac{d^2 U_2}{du^2} + 2U_1 = 0,$$

we find

$$y = (au + b)v + \left(-\frac{au^3}{3} - bu^2 + cu + d \right),$$

where a, b, c and d are arbitrary constants. Therefore the equation of the surface S is

$$3y_4 (y_2 y_3 - y_1 y_4) - 2y_3^3 = 0,$$

which is the equation of a Cayley cubic scroll. If we notice that the conditions imposed upon equations (37) in this case are precisely that *the invariants θ', C', and h shall vanish, we have a simple direct proof of the theorem** *that* "If these invariants vanish identically, the surface must be a Cayley cubic scroll." By the classical method we find the line $y_3 = y_4 = 0$ is the nodal line, and the point $P(1, 0, 0, 0)$ is the pinch-point.

In his third memoir,† Wilczynski has proved an interesting and important theorem for the case when θ' and a certain invariant $\mathfrak{B}$‡ are each different from zero. Now for the surfaces of the type we are considering [i. e., $\Omega' = 0$] the osculating linear complexes are indeterminate, and the invariant $\mathfrak{B}$ vanishes whether θ' vanishes or not. The theorem just referred to undergoes extensive modifications in this case. In fact, we can find the locus of the pinch-points and singular tangent planes not only of the ∞^1 osculating Cayley cubic scrolls belonging to a single generator g of S but also of all the ∞^2 Cayley cubic scrolls associated with S.

An arbitrary point P_Y on the generator g of S will be represented by

* M_3, p. 307.

† M_3, p. 315.

‡ $\mathfrak{B} = -2^6 ((b\theta'_u - 2b_u\theta')^2 - 4b^2 c'^2 \theta')$.

$$Y = y + l\rho,$$

where l is an arbitrary constant.

Since the invariant Ω' vanishes, we may assume that the equations of S have been reduced to the form (37). We shall find

$$Y_{uu} + 2Y_v + fY = 0, \qquad Y_{vv} - aY = 0. \tag{38}$$

The semi-covariants of this system of equations are

$$Y = y + l\rho, \qquad Z = z + l\sigma, \qquad P = aly + \rho, \qquad \Sigma = alz + \sigma.$$

The pinch-point π of the Cayley cubic scroll, osculating S at the point P_Y, is given by the expression

$$\pi = 2^3\Sigma = 2^3(alz + \sigma).$$

Hence as P_Y moves along the generator g, the pinch-point of the osculating Cayley cubic scroll moves along the line joining the semi-covariant points P_z and P_σ. We shall now set up the equations of S in the form (1). They are found to be

$$Y_{uu} + fy + 4\rho = 0, \qquad \rho_{uu} + 4ay + f\rho = 0. \tag{39}$$

The semi-covariants of this system are

$$r = 2z, \qquad s = 2\sigma, \qquad y = y, \qquad \rho = \rho.$$

For uniformity we shall use the notation of the theory of ruled surfaces and write these equations

$$y_{uu} + fy + 4z = 0, \qquad z_{uu} + 4ay + fz = 0, \tag{39a}$$

and denote their semi-covariants by y, z, ρ, σ. If we denote the semi-covariants of (37) by y', z', ρ', σ' we shall find

$$y' = y, \qquad z' = \frac{\rho}{2}, \qquad \rho' = z, \qquad \sigma' = \frac{\sigma}{2}.$$

Therefore the equations of transformation between the coördinate systems determined by $P_{y'}\,P_{z'}\,P_{\rho'}\,P_{\sigma'}$ and $P_y\,P_z\,P_\rho\,P_\sigma$ are

$$\begin{aligned} &\omega' x_1' = x_1, \qquad \omega' x_2' = 2x_3, \qquad \omega' x_3' = x_2, \qquad \omega' x_4' = 2x_4, \\ &\omega x_1 = 2x_1', \qquad \omega x_2 = 2x_3', \qquad \omega x_3 = x_2', \qquad \omega x_4 = x_4', \end{aligned} \tag{40}$$

where ω and ω' are factors of proportionality. From these equations we find that the coördinates of the pinch-point π of the Cayley cubic scroll osculating S at P_y are

$$x_1 = x_2 = 0, \qquad x_3 : x_4 = a,$$

the coördinate system being that determined by $P_y P_z P_\rho P_\sigma$. Equations (37) show that the invariant θ' is a constant; so that the first derived ruled surface of S is the principal surface* of the flecnode congruence of S. We therefore have the theorem:

The locus of the pinch-points of the osculating Cayley cubic scrolls of a ruled surface whose asymptotic curves belong to linear complexes is the principal surface of the flecnode congruence of the surface itself.

An interesting special case arises when $f(u)$ reduces to a constant. It follows from the general theory of ruled surfaces that the surface S then gives rise to a surface of pinch-points S_1 which is a projective transformation of S; and S_1 gives rise to a surface of pinch-points which is the surface S itself. As P_Y describes an asymptotic curve on S, the corresponding pinch-point describes an asymptotic curve on the surface of pinch-points.

From equations (40) it follows that the nodal line of the Cayley cubic scroll of the point P_Y intersects the generator g in the point

$$Q \equiv at_1 y + t_2 z,$$

where $t_1 : t_2 = l$. The flecnodes on g are the points

$$\eta = \sqrt{ay} - z, \qquad \zeta = \sqrt{ay} + z.$$

The cross ratio of the points (P, Q, η, ζ) is -1. The singular tangent plane of the Cayley cubic scroll is tangent to the osculating hyperboloid at the pinch-point.† Its coördinates are

$$(t_2, \ -at_1, \ 0, \ 0).$$

If we denote that generator of S_1 (the surface of pinch-points), which corresponds to g, by g', we see that the singular tangent planes of the Cayley cubic scrolls of g form a pencil whose axis is g'.

Therefore *as the point P moves along the generator g of S, the singular tangent planes of the osculating Cayley cubic scrolls rotate about the corresponding generator of the locus of pinch-points as axis. If $f(u)$ reduces to a constant the point P, the point Q where the nodal line of the Cayley cubic scroll of P intersects the generator g, and the flecnodes η and ζ on g form a harmonic group.*

§ 7. The directrix curves on the surfaces S of the problem can be determined by quadratures

The linear complexes determined by the asymptotic curves through P_y have a congruence in common. The directrix d (the directrix of the first kind)

* W, pp. 216, 218, 220.

† M_3, p. 303.

of this congruence is situated in the tangent plane of P_y; the directrix d' (the directrix of the second kind) passes through P_y. As P_y moves over S, d and d' generate the directrix congruences of the first and second kinds respectively.* The differential equation of the directrix curves on S, which correspond to the developables of these congruences, is

$$bLdu^2 + 2Mdudv - a'Ndv^2 = 0, \tag{41}$$

where

$$L = -2a'(2a'bf + 2a'bb_v + ba'_{uu}) + ba'^2_u = \frac{\theta}{64},$$

$$M = 2a'b(a'b_{uv} - ba'_{uv}) + 2(b^2 a'_u a'_v - a'^2 b_u b_v),$$

$$N = -2b(2a'bg + 2a'ba'_u + a'b_{vv}) + a'b_v^2 = \frac{\theta'}{64}.$$

Since Γ' and Γ'' belong to linear complexes, we have

$$\Omega' = \Omega'' = 0,$$

and therefore

$$M = 2[a'^2(bb_{uv} - b_u b_v) - b^2(a'a'_{uv} - a'_u a'_v)] = 0.$$

Hence equation (41) becomes

$$b\theta du^2 - a'\theta' dv^2 = 0. \tag{41a}$$

If we introduce the transformation†

$$\bar{u} = \alpha(u) = \int \sqrt[3]{\frac{\theta_1(u)}{\varphi_1(u)}}\,du, \qquad \bar{v} = \beta(v) = \int \sqrt[3]{\frac{\theta_2(v)}{\varphi_2(v)}}\,dv, \qquad y = \frac{1}{\sqrt{\alpha_u \beta_v}}\bar{y},$$

we see, from equations (17), (18), (19) and a previous theorem, that (41a) becomes

$$d\bar{u}^2 - d\bar{v}^2 = 0,$$

which gives (on integration)

$$\bar{u} + \bar{v} = \text{const.}, \qquad \bar{u} - \bar{v} = \text{const.},$$

as the finite equations of the two families of directrix curves on S. *Since M vanishes the directrix curves form a conjugate system on the surface S.*

If Ω', Ω'', θ, θ' all vanish (i. e., if the normal forms of the seminvariants f and g are characterized by the vanishing of the constants a, b, c), equation (41) *is satisfied for every direction $du : dv$.* In this case we find that the point

* M_2, p. 114 et seq.

† The origin of this transformation is clear if we recall that $b/a' = \theta_1(u)/\theta_2(v)$, $\theta/a'^2 = \varphi_1(u)$, $\theta'/b^2 = \varphi_2(v)$.

$$P = \lambda (y + y_u\, du + y_v\, dv) + \mu (\tau + \tau_u\, du + \tau_v\, dv)^*$$

is on the line

$$2bx_2 - b_v x_4 = 0, \qquad 2a' x_3 + a'_u x_4 = 0$$

for every value of $du : dv$, if

$$\lambda : \mu = -(2bQ + b_v S) : 2b = -(2a' R + a'_u S') : 2a'.$$

It follows that under these circumstances all the directrices of the second kind in the vicinity of a point P have a point in common. Referred to T the coördinates of this point are found to be

$$(42) \quad x_1 = -(2bQ + b_v S), \quad x_2 = -2a' bb_v, \quad x_3 = -2a'_u b^2, \quad x_4 = 4a' b^2.$$

Using the fact that the invariants Ω' and Ω'' vanish, we find

$$x_1 : x_2 : x_3 : x_4 = 2^3 a'^2 b^2 - a'_u b_v : 2a' b_v : 2a'_u b : -4a' b.$$

To prove that this point is fixed in space we consider the following expression

$$\pi = -(2bQ + b_v S)y - 2a' bb_v z - 2a'_u b^2 \rho + 4a' b^2 \sigma$$

for the point (42), and calculate

$$\pi + \frac{\partial \pi}{\partial u} du + \frac{\partial \pi}{\partial v} dv,$$

making use of the relations

$$\Omega' = \Omega'' = \theta = \theta' = 0.$$

We find

$$\pi + \frac{\partial \pi}{\partial u} du + \frac{\partial \pi}{\partial v} dv = \left(1 + \frac{a_u b + 4a' b_u}{2a' b} du + \frac{a' b_v + 4a'_v b}{2a' b} dv\right)\pi,$$

which shows that π is a fixed point.

Let us now consider the directrices of the first kind. The point

$$P = \lambda (r + r_u\, du + r_v\, dv) + \mu (s + s_u\, du + s_v\, dv)$$

will be on the line joining

$$r = -a'_u y + 2a' z \quad \text{and} \quad s = -b_v y + 2b\rho,$$

if the equations

$$(8a'^2 b^2 + a'_u b_v)(\lambda a'\, dv + \mu b du) = 0, \qquad \lambda a'\, dv + \mu b du = 0,$$

are satisfied. If we choose

$$\lambda : \mu = -b\, du : a'\, dv,$$

---

* $\tau = -a' b_v z - ba'_u \rho + 2a' b\sigma$, $Q = 8a'^2 b^2 - a' b_{uv} - a'_u b_v$, $S = a'_u b + 2a' b_u$, $S' = a' b_v + 2a'_u b$, $R = -(2a' bf + 2a' bb_v + ba'_{uu} + a'_u b_u)$.

we see that two consecutive directrices intersect for every value of the ratio $du : dv$. It therefore follows that all the directrices of the first kind in the vicinity of the point P_y lie in the same plane. We proceed to determine this plane. By using the conditions

$$\Omega' = \Omega'' = \theta = \theta' = 0$$

we can show that the four points

$$r,\ s,\ r + r_u\,du + r_v\,dv,\ s + s_u\,du + s_v\,dv$$

are coplanar. After some reductions we find the equation of their plane, which we shall call the *directrix plane*, to be

$$2^2\,a'\,bx_1 + 2a'_u\,bx_2 + 2a'\,b_v\,x_3 + (2^3\,a'^2\,b^2 + a'_u\,b_v)\,x_4 = 0.$$

It can be readily shown that this *directrix plane* is the null-plane of π in both of the complexes C' and C''. If we introduce the transformation*

$$\frac{\lambda\mu}{k}x_1 = -\alpha\mu X - \lambda\beta Y + \alpha\beta Z + \lambda\mu\omega, \qquad \frac{\lambda\mu}{k}x_3 = \lambda Y - \alpha Z,$$

$$\frac{\lambda\mu}{k}x_2 = \mu X - \beta Z, \qquad \frac{\lambda\mu}{k}x_4 = Z,$$

which refers the plane to the canonical tetrahedron of the point P_y, the above equation becomes

$$Z + 8\omega = 0,$$

and the point π referred to this tetrahedron becomes

$$X = Y = 0, \qquad Z = -2^3, \qquad \omega = 1.$$

These results may be combined in the following theorem:

If both families of asymptotic curves of the surface S belong to linear complexes and if the invariants θ and θ' vanish, the directrix congruences degenerate. The directrix congruence of the first kind consists of the net of lines in the directrix plane, while the directrix congruence of the second kind consists of the sheaf of lines through the null-point of the directrix plane in the complexes C' and C''.

There remains to be considered the case of a degenerate *directrix quadric*. This case is closely allied to that of the degenerate directrix congruence referred to above. There also exist a number of other curves and congruences organically connected with the configurations of this problem. The discussion of all of these questions will be reserved for a future occasion.

THE UNIVERSITY OF CHICAGO.

* M_2, p. 104.

RELATIVELY UNIFORM CONVERGENCE OF SEQUENCES OF FUNCTIONS*

BY

E. W. CHITTENDEN

E. H. Moore† has introduced the notion of uniform convergence of a sequence of functions relative to a scale function. It is the purpose of this paper to study this type of convergence in the field of functions of a real variable.

1. The following definition of relatively uniform convergence is, for the case of a sequence of functions of a real variable, equivalent to the definition given by Moore.

A sequence $\{ \mu_n \}$ of single-valued, real-valued functions μ_n of a variable p, ranging over a set $\mathfrak{P}$ of real elements p, converges *relatively uniformly* on $\mathfrak{P}$ in case there exist functions θ and σ, defined on $\mathfrak{P}$, and for every m an integer n_m (dependent on m), such that for every n greater than or equal to n_m the inequality

$$m \mid \theta - \mu_n \mid \leqq \mid \sigma \mid \ddagger \tag{1}$$

holds for every element p of $\mathfrak{P}$.

The function σ of the definition above is called a *scale function*. The sequence $\{ \mu_n \}$ is said to converge *uniformly relative to the scale function* σ; or more simply, *relatively uniformly*.

2. The following propositions are immediate consequences of the definition of relatively uniform convergence.§

I. Uniformity of convergence relative to a constant scale function different from zero is equivalent to uniform convergence.

II. Uniformity relative to σ implies uniformity relative to every function τ such that $\mid \tau \mid \overline{\leqq} \mid \sigma \mid$.

---

* Presented to the Society March 22, 1913.

† *An Introduction to a Form of General Analysis, The New Haven Mathematical Colloquium*, Yale University Press, New Haven, 1910. This memoir will be cited as I. G. A.

‡ The relation (1) holds identically in p. In such cases we follow the usage of Moore (cf. I. G. A., p. 27) and omit the variable.

§ Cf. I. G. A., p. 33, et seq. All propositions and theorems of this paper are stated for sequences of functions. The corresponding propositions for series of functions are readily inferred.

III. Uniformity as to a function σ such that $A \leqq |\sigma| \leqq B$, where A and B are positive, implies uniform convergence.

IV. If a sequence converges uniformly relative to a scale function σ, but does not converge uniformly, then σ is not bounded.

V. If a sequence of functions is defined on a class $\mathfrak{P}$ and if $\mathfrak{P}$ may be divided into a finite number of parts such that the sequence converges relatively uniformly on each part, then the sequence converges relatively uniformly on $\mathfrak{P}$.*

VI. If a sequence converges relatively uniformly on $\mathfrak{P}$, $\mathfrak{P}$ may be divided into a sequence $\{ \mathfrak{P}_n \}$ such that no two sets $\mathfrak{P}_{n_1}$, $\mathfrak{P}_{n_2}$ $(n_1 \neq n_2)$ have a common element, and such that on each $\mathfrak{P}_n$ the sequence converges uniformly.

3. The following examples are illustrative of relative uniformity of convergence.

(*a*) The class $\mathfrak{P}$ is the interval $0 \leqq p \leqq 1$; the sequence $\{ \mu_n \}$ such that $\mu_n(p) = 1/np$ $(p \neq 0)$, $\mu_n(0) = 0$. This sequence does not converge uniformly, but the function $\sigma(p) = 1/p$ is effective as a scale function.

(*b*) The class $\mathfrak{P}$ is the infinite segment, $1 \leqq p$; $\mu_n(p) = 1/np$. The sequence converges uniformly, but satisfies the stronger condition of uniform convergence relative to the scale function $\sigma(p) = 1/p$.†

4. Using the notation of § 1, we denote by $\phi_n(p)$ the least upper bound of $|\theta(p) - \mu_{n'}(p)|$ for all $n' \geqq n$. If the sequence $\{ \mu_n \}$ converges to θ uniformly, relative to a scale function $\sigma(p)$, then for every m there exists an n_m such that the inequality (1) is satisfied for all $n \geqq n_m$. Hence we may write, in view of the definition of $\phi_n(p)$,

$$m\phi_{n_m}(p) \leqq |\sigma(p)|. \tag{2}$$

As an immediate consequence of this result we have the following theorem:

THEOREM 1. *A necessary and sufficient condition that a sequence* $\{ \mu_n \}$ *of functions* μ_n *converges relatively uniformly on* $\mathfrak{P}$ *to a limit function* θ *is that there exist a sequence* $\{ n_m \}$ *of positive integers such that the sequence* $m\phi_{n_m}(p)$ *has an upper bound* $B(p)$ *for every* p.

5. Let $\mathfrak{P}$ be any interval (a, b), and (r, q) a sub-interval of $\mathfrak{P}$ in which the only point of non-uniform convergence‡ of a sequence $\{ \mu_n \}$ is the end point q. The sequence $\{ \mu_n \}$ being supposed convergent on (r, q), we have a function $A(p) \geqq 1$ such that for every p in (r, q)

* Cf. § 6 of this paper, proposition VIII, which is an extension of proposition V and the converse of VI.

† Other examples are given later in this paper. Cf. also I. G. A.

‡ A point q is a point of non-uniform convergence in case the measure of non-uniformity of convergence of the sequence is greater than zero at q. Cf. W. H. Young, Proceedings of the London Mathematical Society, ser. 2, vol. I (1903–4) p. 91; also E. W. Hobson, *Theory of Functions of a Real Variable*, Cambridge University Press, p. 474, § 342; p. 484, § 349.

$$|\theta(p) - \mu_n(p)| \leqq A(p).$$

We will now show that the sequence $\{\mu_n\}$ converges uniformly on (r, q) relative to the scale function $\sigma(p) = A(p)/(q-p)$; $\sigma(q) = A(q)$. For each m greater than $1/(q-p)$ choose p_m so that $q - p_m = 1/m$. Then in the interval (r, p_m), which contains no point of non-uniformity, the sequence $\{\mu_n\}$ converges uniformly, and therefore n_m exists so that for every $n \geqq n_m$, p in (r, p_m), and $p = q$,

$$m|\theta(p) - \mu_n(p)| \leqq 1.$$

But for every n, and p in the segment $p_m \leqq p < q$,

$$m|\theta(p) - \mu_n(p)| \leqq A(p)/(q-p).$$

The combination of the last two inequalities gives the desired convergence relative to σ.

The result just obtained is stated in the following theorem:

THEOREM 2. *If q is on the left an isolated point of non-uniformity of convergence of a convergent sequence of functions, there exists a left neighborhood of q on which the sequence converges relatively uniformly.*

A similar statement holds if q is isolated on the right.

6. We have occasion to use the following two propositions from general analysis.*

VII. If a sequence of functions is defined on an enumerable set $\mathfrak{P}$ and converges on $\mathfrak{P}$ it converges relatively uniformly on $\mathfrak{P}$.

VIII. If a sequence of functions is defined and converges relatively uniformly on each of a sequence of classes $\mathfrak{P}_n$ it converges relatively uniformly on the class $\mathfrak{P}$ of all elements which are in some class $\mathfrak{P}_n$, the least common superclass of the classes $\mathfrak{P}_n$.

7. Denote by $\mathfrak{Q}$ the set of all points q of non-uniform convergene of a sequence of functions converging on an interval $\mathfrak{P}$ to a definite limit function; by $\mathfrak{Q}'$ the derived set of $\mathfrak{Q}$, and by $\mathfrak{Q}^\circ$ the aggregate of the points of $\mathfrak{Q}$ and $\mathfrak{Q}'$. $\mathfrak{Q}^\circ$ is closed. The set complementary to $\mathfrak{Q}^\circ$ consists of the interior points of an enumerable set of non-overlapping intervals, in each of which the sequence of functions converges relatively uniformly (Theorem 2). It follows from proposition VIII that the sequence converges relatively uniformly on the complement of $\mathfrak{Q}^\circ$. Since the points of $\mathfrak{Q}^\circ$ which are not in $\mathfrak{Q}'$ form an enumerable set, we may state, in view of the preceding remarks and propositions V and VII, the following theorem:

THEOREM 3. *A necessary and sufficient condition that a sequence of functions converges relatively uniformly on an interval $\mathfrak{P}$ is that it converges relatively uni-*

* Proposition VII is from I. G. A., p. 87. A simple proof may be given following the lines of the proof of Theorem 2 above. Proposition VIII is an immediate extension of VII.

formly on the derived set $\mathfrak{Q}'$ of the set $\mathfrak{Q}$ of the points of $\mathfrak{P}$ which are points of non-uniformity.

COROLLARY I. *If $\mathfrak{Q}'$ is enumerable the sequence converges relatively uniformly on $\mathfrak{P}$.*

COROLLARY II. *If a sequence of functions does not converge relatively uniformly on $\mathfrak{P}$, the corresponding set $\mathfrak{Q}'$ is not enumerable, that is, $\mathfrak{Q}$ is dense on a perfect set.*

8. The converse of corollary II above is not true, as is shown by the following example.* $\mathfrak{P}$ is the interval $(0, 1)$. We define the sequence $\{ \mu_n \}$ as follows: $\mu_n(p) = 0$, if p is irrational, zero, or equal to m/k (m and k relatively prime, and $k \neq n$); $\mu_n(p) = 1$, if $p = m/n$ (m, n relatively prime). For this sequence of functions every p is a point of non-uniformity of convergence. However, the functions of the sequence are all zero on the irrational points. Hence by propositions V and VII the sequence converges relatively uniformly.

Osgood† has given an example of a sequence of *continuous* functions converging to zero as a limit for which the set $\mathfrak{Q}$ and its derivative form a perfect set, but which converges relatively uniformly because the sequence is identically zero on $\mathfrak{Q}$ and $\mathfrak{Q}'$.

9. For sequences of continuous functions we have the following theorem:

THEOREM 4. *If a sequence of continuous functions converges on an interval $\mathfrak{P}$ to a continuous limit in such a way that the set $\mathfrak{Q}$ of all points of non-uniformity of convergence is dense on $\mathfrak{P}$, the sequence does not converge relatively uniformly.*

To establish this theorem it is sufficient to show that for every sequence $\{ n_m \}$ of positive integers n_m there exists a p determined by the sequence $\{ n_m \}$ such that the sequence $\{ m\phi_{n_m}(p) \}$‡ is not bounded. Theorem 4 follows in view of Theorem 1.

Let $\{ n_m \}$ be any sequence of integers with index m, and $\{ \mu_n \}$ denote the sequence of continuous functions of the theorem, converging to a continuous limit θ. Since the sequence $\{ \mu_n \}$ does not converge uniformly in any subinterval of $\mathfrak{P}$ there exists an integer m_1 with the following property: for every N there exists an n_1 greater than N and a p_1 (interior to $\mathfrak{P}$) such that

$$m_1 | \theta(p_1) - \mu_{n_1}(p_1) | > 1.$$

Since θ and μ_{n_1} are continuous, we may find an interval I_1 interior to $\mathfrak{P}$ such that for every p in I_1

$$m_1 | \theta(p) - \mu_{n_1}(p) | > 1.$$

*Cf. W. H. Young, Proceedings of the London Mathematical Society, Ser. 2, vol. I (1903–4), p. 94; also E. W. Hobson, *Theory of Functions of a Real Variable*, p. 487.

† *Non-Uniform Convergence and the Integration of Series Term by Term*, American Journal of Mathematics, vol. 19 (1897), p. 168.

‡ Cf. § 4 for explanation of this notation.

Consequently, if we take N equal to n_{m_1} of the sequence $\{ n_m \}$, we have

$$m_1 \phi_{n_{m_1}}(p) > 1$$

for every p in I_1.

By similar reasoning we may show that there exists an m_2 and an interval I_2 interior to I_1 such for every p in I_2

$$m_2 \phi_{n_{m_2}}(p) > 2.$$

Proceeding step by step in the manner indicated we may obtain a sequence $\{ m_j \}$ and a sequence $\{ I_j \}$ of closed intervals, each interior to the preceding, such that for every j we have on I_j

$$m_j \phi_{n_{m_j}}(p) > j.$$

There must exist an element p° common to the intervals I_j. For such an element we have for every j

$$m_j \phi_{n_{m_j}}(p^\circ) > j.$$

The sequence $m \phi_{n_m}(p^\circ)$ is therefore unbounded, which was to be proved.

Osgood* has given an example of a sequence of continuous functions converging to a continuous limit for which the set $\mathfrak{Q}$ is dense on $\mathfrak{P}$. It follows from the theorem just proved that this sequence does not converge relatively uniformly.

10. A point of non-uniformity of convergence of a sequence of functions relative to a subset $\mathfrak{R}$ of $\mathfrak{P}$ is a point whose measure of non-uniformity of convergence relative to $\mathfrak{R}$ is greater than zero.†

Evidently a point of non-uniformity of convergence relative to $\mathfrak{R}$ is a point of non-uniformity of convergence relative to any set containing $\mathfrak{R}$, and therefore of $\mathfrak{P}$.

A slight extension of the proof of Theorem 4 serves to establish the following generalization of that theorem.

THEOREM 5. *If a sequence of functions defined on $\mathfrak{P}$ and continuous on $\mathfrak{R}$, a perfect subset of $\mathfrak{P}$, converges on $\mathfrak{P}$ to a limit which is continuous on $\mathfrak{R}$ in such a fashion that the class $\mathfrak{Q}$ of all points of $\mathfrak{R}$ which are points of non-uniformity of convergence relative to $\mathfrak{R}$ is dense on $\mathfrak{R}$, then the sequence of functions does not converge relatively uniformly on $\mathfrak{R}$ or on $\mathfrak{P}$.*

URBANA, ILL.,
May 7, 1913.

* Osgood, loc. cit., p. 171.

† Hobson, loc. cit., p. 484, § 349.

NOTE ON FERMAT'S LAST THEOREM*

BY

H. S. VANDIVER

1. If x, y and z are integers prime to each other, and

$$x^p + y^p + z^p = 0, \tag{1}$$

where p is a prime, and

$$q(r) = \frac{r^{p-1} - 1}{p},$$

Furtwängler† has shown that

$$q(r) \equiv 0 \pmod{p}$$

for each factor r of x, in case $x \not\equiv 0 \pmod{p}$, and for each factor r of $x^2 - y^2$, in case $x^2 - y^2$ is prime to p.

By applying this theorem, Furtwängler deduces the criterion of Wieferich $q(2) \equiv 0 \pmod{p}$ and the criterion of Mirimanoff $q(3) \equiv 0 \pmod{p}$ for the solution of (1) in integers prime to p. I shall here extend these results and show that in addition we have, provided that $q(2) \not\equiv 0 \pmod{p^3}$, the criteria $q(5) \equiv 0 \pmod{p}$ for $p \equiv 1 \pmod{3}$ and $q(5) \equiv q(7) \equiv 0 \pmod{p}$ for $p \equiv 2 \pmod{3}$.

2. Assume that x, y and z are prime to each other and to p and that $p > 5$. If one of the integers x, y, z is divisible by 5, then $q(5) \equiv 0 \pmod{p}$ by Furtwängler's theorem. If none of them is so divisible, then, modulo 5, x^p, y^p, z^p have the residues ± 2, ± 2, ± 1 or ± 1, ± 1, ∓ 2 in some order. We may therefore take $x^p \equiv y^p \pmod{5}$. Then $x \equiv y \pmod{5}$, since every integer has a unique cube root modulo 5. Thus 5 is a divisor of $x^2 - y^2$. Hence (§ 1) $q(5) \equiv 0 \pmod{p}$, unless $x^2 \equiv y^2 \pmod{p}$, i. e., unless $x \equiv y \pmod{p}$, since $x \equiv -y$ and $x + y + z \equiv 0 \pmod{p}$ would imply $z \equiv 0 \pmod{p}$, contrary to hypothesis. Using $x + y + z \equiv 0$, we may state the result:

If (1) *is satisfied by integers prime to* p, *then the congruence*

$$q(5)(t-1)(t+2)(t+1/2) \equiv 0 \pmod{p} \tag{2}$$

is satisfied by each of the following values of t:

$$\frac{x}{y}, \frac{y}{x}, \frac{x}{z}, \frac{z}{x}, \frac{y}{z}, \frac{z}{y}. \tag{3}$$

* Presented to the Society, February 28, 1914.

† Sitzungsberichte K. Akademie der Wissenschaften, Wien, vol. 121 (1912), p. 589.

3. From (1) we have

$$\frac{x^p + y^p}{x + y} = v^p,$$

when v is an integer, since the quotient is relatively prime to $x + y$ and hence is a pth power. Since v is a factor of z, it is not divisible by p, and is of the form $1 + kp$, since the fraction is congruent to $-z^p/(-z)$ modulo p. Furthermore,

$$(1 + kp)^{p-1} \equiv 1 \pmod{p^2}$$

by Furtwängler's theorem. Multiply the members by $1 + kp$ and apply $(1 + kp)^p \equiv 1 \pmod{p^2}$. Hence $k \equiv 0 \pmod{p}$, and $v^p \equiv 1 \pmod{p^3}$. Hence

$$x^p + y^p \equiv x + y,$$

$$x^p + z^p \equiv x + z, \pmod{p^3}, \tag{4}$$

$$y^p + z^p \equiv y + z$$

$$x^p \equiv x, \quad y^p \equiv y, \quad z^p \equiv z \pmod{p^3}. \tag{5}$$

Hence by (1),

$$x + y + z \equiv 0 \pmod{p^3}. \tag{6}$$

4. Suppose that $y = x + p\mu$. Substituting in the first relation (4), we have

$$x^p + (x + p\mu)^p \equiv 2x + p\mu \pmod{p^3},$$

$$2x^p + p^2 \mu x^{p-1} \equiv 2x + p\mu \pmod{p^3}.$$

Hence, by (5),

$$p\mu(px^{p-1} - 1) \equiv 0 \pmod{p^3}, \qquad \mu \equiv 0 \pmod{p^2}.$$

We may therefore set $y = x + p^3\mu$. Then, from (6), $z = -2x + p^3\nu$. Hence, from (1),

$$x^p + (x + p^3\mu)^p + (-2x + p^3\nu)^p = 0,$$

$$2x^p - 2^p x^p \equiv 0 \pmod{p^4},$$

$$q(2) \equiv 0 \pmod{p^3}.$$

5. Now consider the criteria given by Mirimanoff* for the solution of (1). He showed that if (1) is satisfied by integers prime to p, then the ratios (3) satisfy

$$F(t) = \prod_{i=1}^{m-1}(t + \alpha^i) \sum_{i=1}^{m-1} \frac{R_i}{t + \alpha^i} \equiv 0 \pmod{p} \tag{7}$$

when $m = 2, 3, \cdots, p - 1$ and

$$R_i = \frac{\varphi_{p-1}(-\alpha^i)}{(1 - \alpha^i)^{p-1}}, \qquad \alpha = e^{2\pi\sqrt{-1}/m},$$

*Journal für Mathematik, vol. 139 (1911), p. 309 et seq.

$$\varphi_i(t) = t - 2^{i-1}t^2 + 3^{i-1}t^3 - \cdots - (p-1)^{i-1}t^{p-1}.$$

He also showed that

$$F(-1) \equiv (-1)^m m\, q(m) \pmod{p}. \tag{8}$$

Let $m = 7$ in (7). Assume $p > 7$. The resulting congruence is of degree 5 in t. The ratios (3) have 6 incongruent values unless one of them is a root of

$$(t-1)(t+2)(t+1/2) \equiv 0 \text{ or } t^2 + t + 1 \equiv 0 \pmod{p}.$$

If $p \equiv 2 \pmod 3$, the latter is not possible for t rational. Hence $t \equiv 1$, -2 or $-1/2$ and therefore, by § 4, $q(2) \equiv 0 \pmod{p^3}$ unless (7) is an identity. In the latter case we may set $t \equiv -1$ and obtain $q(7) \equiv 0 \pmod p$ by reason of (8). Hence the criteria:

If (1) *is satisfied by integers prime to* p, *then either*

$$q(2) \equiv 0 \pmod{p^3}, \qquad q(3) \equiv 0 \pmod p,$$

or else

$$q(2) \equiv q(3) \equiv q(5) \equiv 0 \pmod p;$$

and if $p \equiv 2 \pmod 3$,

$$q(7) \equiv 0 \pmod p.$$

6. There are no primes p at present known such that $q(2) \equiv 0 \pmod{p^3}$. Meissner* observes that $q(2) \equiv 0 \pmod{1{,}093}$, but finds $q(2) \not\equiv 0 \pmod{1{,}093^2}$. He also states that $q(2) \not\equiv 0 \pmod p$ for every $p < 2{,}000$ excepting 1,093.

7. If any one of the forms

$$2^\alpha 3^\beta \pm 1, \qquad 2^\alpha \pm 3^\beta,$$

where α and β are positive integers or zero, is divisible by a prime p but is not divisible by p^2, then p is excluded as an exponent in (1), if x, y and z are prime to each other and to p.† For, if p is admissible in (1), then $q(2) \equiv q(3) \equiv 0 \pmod p$, and

$$(2^\alpha)^{p-1} \equiv (3^\beta)^{p-1} \equiv 1, \qquad (2^\alpha 3^\beta)^{p-1} \equiv 1 \pmod{p^2}. \tag{9}$$

But if $2^\alpha 3^\beta \pm 1 \equiv 0$ or $2^\alpha \pm 3^\beta \equiv 0 \pmod p$ but $\not\equiv 0 \pmod{p^2}$, then

$$(2^\alpha 3^\beta)^{p-1} \not\equiv 1 \pmod{p^2}, \qquad (2^\alpha)^{p-1} \not\equiv (3^\beta)^{p-1} \pmod{p^2},$$

which contradict (9). As an example, the integer $p = 2^{61} - 1$ is known to be prime, hence it is excluded as an exponent in (1).

*Sitzungsberichte der Preuss. Akademie der Wissenschaften, 1913, no. 35, p. 663.

†See also Mirimanoff, Paris Comptes Rendus, vol. 150 (1910), p. 206.

A SET OF AXIOMS FOR LINE GEOMETRY*

BY

E. R. HEDRICK AND LOUIS INGOLD

1. Introduction. This paper proposes a set of axioms for line geometry† based upon the *line* as an undefined element and *intersection* as an undefined relation between unordered pairs of elements. These same undefined concepts have been used in a set of axioms by Pieri.‡

The Pieri system, however, does not seem to accomplish all that could be desired in the way of simplicity, and there are several rather obvious redundancies in it which the authors of the present paper have been able to avoid.§

* Presented to the Society in a somewhat different form, December 1, 1911. At that time the paper of Pieri mentioned below was unknown to the authors.

† The precise types of geometry studied are discussed below.

‡ *Sui principi che regono la geometria delle rette*, Atti della Accademia di Torino, vol. 36 (1901), pp. 335–351. The relation used by Pieri is introduced first as a relation between ordered pairs of elements; but he states an axiom later to make the relation independent of the order.

§ For reference, we give here the system of Pieri:

Postulº I. Il raggio e una classe non illusoria.

Postulº II. Ogni raggia incontra se stesso.

Postulº III. Essendo a un raggio, esiste almeno un raggio, che taglia a senza coincider con a.

Postulº IV. Posto che a, b siano raggi distinti, e che a tagli b, si deduce che b taglia a.

Df. Date le rette a, b che s'incontrino senza confondersi, per fascio ab, s'intende la classe di tutti quei raggi, ognuno de quali s'incontra con ogni retta, che tagli al medesimo tempo a e b. La stessa figura si rappresenta col simbolo (ab).

Postulº V. Dal supposto, che due raggi a e b s'incontrino senza confondersi, e c sia un raggio di (ab) non coincidente con a, si deduce che b appartiene ad (ac).

Postulº VI. Qualunque volta due raggi a e b si tagliano senza confondersi, nel fascio (ab) giace almeno un raggio diverso da a e da b.

Postulº VII. Dato che a, b siano raggi distinti e l'un l'altro incidenti, dovra esistere un raggio che li tagli ambedue senza giacere in (ab).

Df. Si chiama triraggio, o trilatispigolo, la figura costituita in tre rette, di cui ciascuna incontri ciascuna sepero non esiste alcun fascio di raggi che le contenga.

Postulº VIII. Posto che i raggi a, b e c siano elementi d'un trilatispigolo, se a' e un (bc) diverso da b e da c, similmente b' un (ca) diverso da c e da a, deve esisteri almeno un raggio comune ai due fasci (aa') e (bb').

Postulº IX. Se di due rette a e b, che s'incontrano senza coincidere, ognuna e tagliata da due rette sghembe c e d; allora ogni raggio che incontri al medisimo tempo a e b, dovra tagliare uno almeno dei raggi c, d.

In this paper a set of five axioms is introduced which are essentially equivalent to Pieri's system, and which seem to be quite as simple in statement and in application as those of Pieri. It is shown also that the resulting space is a *general projective space*, that is, the space is either *properly* or *improperly* projective according to the classification introduced by Veblen and Young.* This follows from the fact that the five axioms of this paper are equivalent to the two groups A and E of assumptions given by them.

The axioms are stated in terms of the undefined concepts *line* and *intersection*, and two defined secondary concepts, viz., *a pencil of lines*, and what we have called *a doublet of lines*. These secondary concepts are defined in terms of line and intersection in what follows. After the axioms, a third secondary concept, *a field of lines*, is defined.

For the purpose of easy orientation, the reader may think of a field of lines either as a set of lines lying in a given projective plane, or as a set of lines passing through a given point of space. He may regard a pencil of lines as a set of lines of a plane through a given point of that plane. A doublet may be thought of as a set consisting of all the lines of a plane together with all the lines of space which pass through a given point of that plane. It should be noticed, however, that the concepts *plane* and *point* are not used in setting up the system; rather we have been able to define these concepts in terms of the fundamental notions *line* and *intersection*. The concepts *field*, *pencil*, and *doublet* seem particularly adapted to a treatment of line geometry and their early introduction makes possible a great simplification of the axioms.

The plan of the paper lends itself to extension to higher dimensions, although the generalization is not immediate. On some other occasion this extension will be treated.

2. The Axioms and their Independence. In what follows, the undefined elements (lines) will be denoted by small Roman letters. We shall presuppose nothing regarding the character of the undefined relation (intersection) except

Postul^{o} X. E ogni raggio, che incontri tutte e quattro le rette a, b, c, d giace nel fascio (ab).

Postul^{o} XI. Dati ancora due raggi a e b che s'incontrino senza coincidere, e date una retta g che non tagli a ne b, sara ,necessario che questa g incontri un qualche raggio di (ab).

Of these, Nos. I, III, V, VI, VII, VIII, IX, X, XI, are consequences of our axioms. No. II is unnecessary in the present system, since the relation of intersection is not defined to apply for a pair of identical lines. No. IV is not necessary in the system unless it is presumed that the relation of intersection depends on the order of the two elements as well as on the elements themselves.

It may be added that Pieri's V and VIII can be proved from his II, IV, VII, IX, X, and XI, and that the major content of his III and his VI can be proved from his XI.

* *A set of assumptions for projective geometry*, American Journal of Mathematics, vol. 30 (1908), pp. 347–380; in particular, p. 352. These axioms are stated below in § 6. See also Veblen and Young, *Projective Geometry*, Ginn & Co., 1910.

that it holds or fails for any two distinct elements.* If the relation holds for two lines a and b we shall say that they *intersect*, or that a *meets* b, or that b meets a. If the relation fails, we shall say that a and b are *skew*, or that they are *a pair of skew lines*.

DEFINITION 1. If a and b are a pair of intersecting lines, the set of lines consisting of a and b together with all the lines that meet both a and b is called a *doublet*, and is denoted by the symbol $[ab]$.

DEFINITION 2. The set of lines of a doublet $[ab]$ each of which meets every other line of the doublet $[ab]$ is called a *pencil*, and is denoted by the symbol (ab).

In any doublet $[a, b]$ there is evidently just one pencil (a, b).

AXIOM I. *There exists at least one pair of intersecting lines.*

AXIOM II. *In every doublet* $[ab]$ *there is at least one line not belonging to the pencil* (ab).

AXIOM III. *Every line* l *of every doublet* $[ab]$ *which is not in the pencil* (ab) *meets one and only one of any two skew lines of* $[ab]$ *both different from* l.

AXIOM IV. *Both of any two intersecting lines are skew to some third line.*†

AXIOM V. *Every line meets at least one line of every pencil.*‡

We give the following examples to show the technical independence of these five axioms, in their order.

1. The elements are two lines; they do not meet. All the axioms except I are satisfied (vacuously).

2. The elements are the lines of one plane together with the five perpendiculars to that plane at the vertices of a certain regular pentagon of the plane. Intersection shall be interpreted as usual (projectively), except that the perpendiculars to the plane shall all be skew to each other. Axiom II is the only axiom violated.

3. The elements are the lines of an ordinary projective space and one extra line l'. Intersection shall be interpreted as usual for all lines except l', while l' shall meet all lines that meet a given line l of space and be skew to all other lines of space, including l itself. Axiom III alone is violated.

4. The elements are the edges of a tetrahedron. Intersection is to be understood in the ordinary sense. All the axioms except IV are satisfied.

* It should be noticed that in this paper the undefined relation holds or fails for the set of objects (a, b), not for the sequence (a, b).

† It is necessary to assume this only for every pair of lines of which at least one belongs to a certain fixed pencil. It seems actually to defeat the desire for simplicity to remove this redundancy, but it may be done by stating IV as follows:

IV′. If there exists a pair of intersecting lines, then there are two intersecting lines a and b such that, if either of any two intersecting lines m and n belongs to (ab), a line k skew to both m and n exists.

‡ It is evident that this last axiom contains a redundancy, in that we already know that any line which meets either a or b meets at least one line of the pencil (ab).

5. The elements are the lines of an ordinary projective space and one extra line l'. Intersection shall have the usual meaning, except that l' shall be skew to every other line. All the axioms except V are satisfied.

Axiom I may be broken into two parts, as follows:

I_1. *There exists at least one line.* (Pieri's I.)

I_2. *If there exists one line, there exists a pair of intersecting lines.**

These statements are independent of each other, and the system I_1, I_2, II, III, IV, V form a mutually independent system. For, the examples (2), (3), (4), (5), above, need not be altered; and (1) serves as an independence example for I_2. If there are no lines whatever, all but I_1 are satisfied.

Again, Axiom I_2 may be divided into two parts, as follows:

I_2'. *If there exists one line, there exists at least one other line.*

I_2''. *If there exist two lines, there exist two intersecting lines.*†

Axiom III can be readily broken into two parts by replacing in it the words *one and only one* by (III_1) the word *one*, or (III_2) the words *not more than one.*

3. Preliminary Definitions and Theorems. Before proceeding to any theorems, we shall define another secondary concept, a *field of lines.*

DEFINITION 3. A set of lines of a doublet is called a *field* if (*a*) the set contains no pair of skew lines; (*b*) every line of the doublet not in the set is skew to at least one line of the set.

Fields will sometimes be denoted by small Greek letters. As an illustration of the usefulness of this concept, we may state, by Axiom III, that the lines of a given doublet $[ab]$ not in a given field α of $[ab]$ are all skew to any line of α not in the pencil (ab).

THEOREM I. *In any doublet there are two and only two fields.*

By Axiom II, there exists at least one pair of skew lines p and q in any given doublet. All the lines of the doublet that meet p meet each other, by Axiom III. Now p is skew to all other lines of the doublet. Hence p together with the lines of the doublet that meet p constitute a field.

* It should be noticed that I_2 is by no means equivalent to Pieri's III. But our V (or Pieri's XI) and I_2 suffice to prove his III.

† The possibility of dividing most axioms into component parts seems not to have been sufficiently emphasized. For example, the statement of A_1 of the Veblen-Young list (§6, below) is rather obviously composed of the two parts:

A_1'. If A and B are distinct points, there is at least one line.

A_1''. If A and B are distinct points and if there is at least one line, there is a line containing both A and B.

It is evident that A_1' is also a portion of E_1. A_1'' can be split into two parts in a similar manner, and so on *ad infinitum*. In fact it seems to be true that the existence of an infinite number of elements in any system at once renders it possible to arrange at least a countably infinite set of independent axioms whose content does not overlap, and which are all provable from any axioms that define that set. Hence it seems difficult to define conveniently the difference between single and multiple statements.

Another field consists of q and all the lines of the given doublet that meet q. There is no other field, since there is no line skew to both p and q, by Axiom III.

THEOREM 2. *If a line p of a doublet $[ab]$ meets two skew lines c and k of that doublet, it belongs to the pencil (ab).*

For, since every pair of the set a, b, p, c intersect, they belong to a field; likewise, a, b, p, k belong to a field. These fields are different, since c and k cannot belong to the same field. Hence p lies in both fields and therefore belongs to the pencil (ab).

COROLLARY. *The pencil (ab) contains the lines a and b.*

THEOREM 3. *If p and q are distinct lines of the pencil (ab), the doublet $[pq]$ is identical with the doublet $[ab]$.*

For, p and q belong to both fields of the doublet $[ab]$; hence a line meeting a and b meets p and q; and therefore the doublet $[ab]$ is contained in the doublet $[pq]$.

Again, a and b belong to $[pq]$ and each meets at least one pair of skew lines that belongs to $[ab]$ and therefore also to $[pq]$; hence, by Theorem 2, a and b belong to the pencil (pq).

Interchanging a and b with p and q in the argument above, it is seen that any line of $[pq]$ is also a line of $[ab]$.

COROLLARY 2. *If a line meets two lines of a pencil, it meets every line of the pencil.*

COROLLARY 3. *If c is a line of the pencil (ab) and $c \neq b$, then a is a line of the pencil (bc).*

THEOREM 4. *If c is a line of the doublet $[ab]$ which does not belong to the pencil (ab), that field of the doublet $[ab]$ which contains c consists of a, b, c and all the lines that meet a, b, and c.*

Let p and q be any two lines that meet a, b, and c; suppose that p and q are skew. Then, by Theorem 2, since c meets p and q, c belongs to the pencil (ab), contrary to hypothesis. Hence p and q intersect and therefore belong to the field of the doublet $[ab]$ which contains c.

DEFINITION 4. If c is a line of the doublet $[ab]$ which does not belong to the pencil (ab), the three lines a, b, c form a *triad*.

THEOREM 5. *There is one and only one field containing the lines of a given triad.*

This follows immediately from Theorem 4 and Definition 2.

It will be convenient to denote the field determined by the lines a, b, c of a triad by (abc).

THEOREM 6. *If two distinct pencils belong to the same field, they have one and only one line in common.*

Let the distinct pencils (ab) and (cd) belong to a field α; then there is a line l meeting a and b but not belonging to α. The lines c and d cannot both

belong to the pencil (ab); for, by Theorem 3, (ab) would then coincide with (cd). Hence l does not meet all lines of (cd). By Axiom V, l meets at least one line m of (cd). Hence by Corollary 2, l meets one and only one line m of (cd). The line m belongs to (ab), since l meets only those lines of α which are lines of (ab). No other line can be common to the two pencils, since l meets every line of (ab) and only the line m of (cd).

THEOREM 7. *The lines of a field that meet a line not belonging to the field form a pencil.*

Let p be a line not belonging to the field α of the doublet $[ab]$. If p meets a and b, then p meets every line of the pencil (ab), and the theorem is proved.

If p does not meet both a and b, there is at least one line c of the pencil (ab) which p meets, by Axiom V; also there is a line l of the field α which does not belong to the pencil (ab). Then α is identical with the field (abl) by Theorem 5. Suppose p does not meet a. The line p meets one line d of the pencil (al). Hence p meets two lines c and d of the field α, so that p meets all lines of the pencil (cd).

There is no other line of the field α which meets p. For if p meets all lines of the pencil (cd) and one other line of the field α, then p belongs to the field α.

4. Classification of Fields. We shall now show that fields fall into two main classes, for which a principle of duality holds. Later we shall identify these two classes of fields with points and planes of ordinary geometry.*

DEFINITION 5. Two distinct fields are said to be *incident* to one another if they have in common all the lines of a pencil.

THEOREM 8. *If two distinct fields are incident to the same field they have in common one and only one line.*

If α and β are both incident to ϕ, there is a pencil P common to α and ϕ and a pencil Q common to β and ϕ.

The pencils P and Q have one and only one line l in common. Let p and q be two other lines belonging to P and Q, respectively. Then $l, p,$ and q are three lines of ϕ forming a triad; also p belongs to α and q belongs to β. Let k be any line common to α and β. Then k meets $l, p,$ and q. Hence k belongs to ϕ, and therefore to each of the pencils P and Q. It follows that k is identical with l.

DEFINITION 6. Two fields which have one and only one line in common are said to be *similar* to each other; otherwise they are not similar, except that a field is said to be similar to itself.

THEOREM 9. *If a field α is not similar to a field ψ and if ψ is incident to a field ϕ, then α is similar to ϕ.*

* Such a classification, whose purpose is to realize the well-known duality of point and plane in line geometry, will obviously be a part of any treatment of the subject. Thus Pieri, *loc. cit.*, gives a treatment similar to that of this paper, but his proofs are of course essentially different.

1. If α is incident to ψ, there is one and only one line common to α and ϕ, by Theorem 8; hence α is similar to ϕ.

2. If α is not incident to ψ, let l be any line of α which does not belong to ϕ (if there is no such line, α and ϕ are identical and therefore similar). Denote by P the pencil common to ϕ and ψ. Then l meets one and only one line m of P; let r be another line of P. The lines of α which meet m form a pencil A; r meets at least one line s of A; s meets m and r; hence s belongs to the field ϕ. For, the line s cannot belong to ψ, since it belongs to α. The lines l, m, and s form a triad; α is incident to the field (lms). The field ϕ also is incident to (lms). Hence α and ϕ have only the line s in common, so that α is similar to ϕ, as was to be proved.

THEOREM 10. *If each of two fields α and β is similar to a field ϕ, α is similar to β.*

Let a be common to α and ϕ, and b common to β and ϕ. If a coincides with b, there is no other line common to α and β. For, suppose k is such a line, so that α and β are incident. Then, since k does not belong to ϕ, k meets ϕ in a pencil A. Let a' be a line of this pencil different from a. Then a, a', and k form a triad. The field $(aa'k)$ contains the pencil (ak) which is common to α and β. Hence $(aa'k)$ is identical with either α or β, and therefore α or β has the pencil $(aa') = A$ in common with ϕ, contrary to hypothesis.

If a is different from b, then at least a meets b, since they belong to the same field ϕ. Let ψ be the field incident to ϕ containing a and b. Then a meets β in a pencil B. Let b' denote a line of B different from b. The line b' meets a and b and does not belong to ϕ, since b' belongs to β. Hence b' belongs to ψ and therefore β has the pencil (bb') in common with ψ. Similarly, α has a pencil in common with ψ. Hence, by Theorem 8, α is similar to β.

THEOREM 11. *If each of two fields α and β is not similar to a field ϕ, α is similar to β.*

Let ψ be a field incident to ϕ. Then, by Theorem 9, α and β are both similar to ψ. Hence by Theorem 10, α is similar to β.

THEOREM 12. *If a field α is similar to a field ϕ and if another field β is not similar to ϕ, then α is not similar to β.*

For, if α were similar to β, then, by Theorem 10, we would have β similar to ϕ, contrary to hypothesis.

By Theorems 10, 11, 12, we can separate all fields into two classes, those in one class being all that are similar to a given field ϕ, and those of the other class being the fields which are not similar to ϕ. Any two belonging to the same class are similar to each other, any two belonging to different classes are not similar to each other.

5. Identification of Fields with Points and Planes. We shall call the fields of one of the classes just mentioned *planes*, and those of the other class *points*. Since these two classes of fields possess a complete duality, a distinction as to which fields are to be called points and which are to be called planes is impossible.

If a point and a plane are incident to one another, we say that the point *lies in* the plane or that the plane *passes through* the point, or *contains* the point.

If a line belongs to a plane, we say that the line *lies in* the plane or that the plane *passes through* the line.

We say that two intersecting lines a and b *meet in a point* and also that they *determine a plane*.

Other expressions commonly used in geometry, such as "the line joining two points," "the line of intersection of two planes," etc., may also be defined in an obvious way.

It is now possible to prove many theorems of projective geometry relating to the intersections of lines and planes and the collinearity of points; for example, Desargues's theorem regarding two perspective triangles could be proved; and, again, the quadrangle construction for the harmonic conjugate of a given point with respect to two others on the same line is clearly possible.

6. Comparison with Other Systems of Axioms. In conclusion we compare the axioms of this paper with a complete system of axioms for projective geometry, so that it may be seen what further axioms are needed to complete the usual theory. The system chosen for this purpose is that of Veblen and Young cited above. The following groups A and E of their assumptions will be proved to be equivalent to our Axioms I–V.

A_1. If A and B are distinct points, there is at least one line containing both A and B.

A_2. If A and B are distinct points, there is not more than one line containing both A and B.

A_3. If A, B, C are points not belonging to the same line, and if a line l contains a point D of a line joining B and C and a point E, distinct from D, of a line joining C and A, then the line l contains a point F of a line joining A and B.

E_0. There are at least three points on every line.

E_1. There exists at least one line.

E_2. It is not true that every point lies on every line.

E_3. It is not true that every point lies on every plane.*

* Under at least one interpretation of these statements, E_1 is a logical consequence of E_2, and both E_1 and E_2 are consequences of E_3. The somewhat altered statements in Veblen and Young, *Projective Geometry*, p. 24, appear to be independent, however.

E_3'. If S is a three-space, every point lies in S.

In this system the plane and the three-space are defined in terms of the undefined elements point and line as follows:

DEFINITION. If P, Q, R are three points not on the same line, and l is a line joining Q and R, the class S_2 of all points such that every point of S_2 is collinear with P and some point of l is called the *plane* determined by P and l. If P, Q, R, T are four points not in the same line or plane, and if α is a plane containing Q, R, and T, the class S_3 of all points such that every point of S_3 is collinear with P and some point of α is called the *three-space* determined by P and α.

Assumptions A_1 and A_2 follow from our axioms by the definition of similar fields.

To prove A_3, let a be common to the (point) fields C and B, and let b and c be the lines common to the (point) fields C and A, and A and B, respectively. Then a, b, and c form a triad (triangle). Let ϕ be the (plane) field determined by a, b, and c. If a line l meets a and b, and if the (point) field D determined by a and l and the (point) field E determined by b and l are distinct, then a, b, and l do not all three belong to a (point) field. Hence l belongs to the (plane) field (abc), so that l meets c, and the (point) field determined by l and c is a point of a line c joining A and B.

E_0 follows from I, II, IV, and V in an obvious manner.

E_1 follows from I; or indeed from E_2, as remarked above.

E_2 follows from E_3; or it may be proved directly as follows. Consider a line l on which there is a point P, and let a and b be two lines passing through P. We shall denote the field which defines the point P by α. The lines a and b determine also a (plane) field β. Let c be a line of this plane not belonging to the pencil (ab). Then c meets both a and b; the three points determined by a and b, b and c, c and a, respectively, do not all three lie on the same line. Hence E_2 is proved.

To prove E_3, consider the plane α of any plane triad a, b, c. There is a line p, not lying in α, which passes through the point P determined by a and b. The lines p and a determine a plane which contains a line q not passing through the point P. The lines p and q determine the point R. Now R is distinct from P, for otherwise q would pass through P; hence R does not lie in α, for then p would also lie in α.

To prove E_3, let S_3 be a three-space determined by a point P and a plane α containing the intersecting lines a and b; and let Q denote a point. Let l be a line common to P and Q. By V, l meets at least one line c of the pencil (ab). Hence Q is collinear with P and the point determined by l and c of the plane α. Therefore Q is a point of S_3 by definition.

It can also be shown that I, II, III, IV, and V follow from the assumptions A and E.

7. Conclusion. In order to complete the entire theory, it is only necessary to follow precisely the steps taken by Veblen and Young, for example, beyond their assumptions A and E, in the paper referred to above, and those taken by Veblen in a later paper.* The remaining assumptions of the Veblen-Young system are evidently independent of our set. Hence there seems to be no essential reason for pursuing the details of this discussion further.

COLUMBIA, MO.,
January, 1914.

* *On the definition of the multiplication of irrational numbers*, American Journal of Mathematics, vol. 34 (1912), pp. 211–214.

THE CAUCHY PROBLEM FOR INTEGRO-DIFFERENTIAL EQUATIONS.*

BY

GRIFFITH C. EVANS

INTRODUCTION

1. A large part of the theory of integral and integro-differential equations may be reduced to the corresponding theory of algebraic and differential equations by the introduction of convenient symbolism. For direct operations this analogy is well expressed by merely considering as a product the combination

$$AB = \int A(r, \tau)\, B(\tau, s)\, d\tau, \tag{1}$$

taking the integral with constant limits a, b, or variable limits s, r, as the case may be.† In this symbolism, however, the vanishing of the "product" of two functions does not imply necessarily that one of the functions vanishes. And therefore for the treatment of the inverse operations and the construction of a complete *algebra*, it is more convenient to consider, instead of the combination written above, the following combination of certain complex quantities. Let

$$\xi = u + jU(r, s), \qquad \eta = v + jV(r, s).$$

By $\xi\eta$ we understand the quantity

$$\xi\eta = uv + j\{uV(r, s) + vU(r, s) + \int U(r, \tau)\, V(\tau, s)\, d\tau\}, \tag{2}$$

taking as the limits of the integral a, b or s, r, according as the algebra is of one kind or the other.‡ The complex unit j is merely used to separate the

* Presented to the Society September, 1913.

† V. Volterra, *Questioni generali sulle equazioni integrali ed integro-differenziali*, Rendiconti della R. Accademia dei Lincei, vol. 19 (1910), p. 169 and p. 178.

‡ In order not to introduce special parameter values into the formal consideration of the problem, it is convenient to make the assumption that for constant limits of integration the j-coefficients contain a parameter λ in such a way that $|U(r, s)| < \lambda M$, and to consider small values of λ. That convention will not generally be necessary here, since we are to introduce explicitly parameters x, y. Where the introduction of such a parameter is necessary it will be specified.

quantity ξ into two parts, which may be called respectively ordinary coefficient, denoted by Ord ξ, and j-coefficient, written j-coefficient ξ. Each coefficient may be complex in the usual sense. By $\xi = 0$ is understood, of course, the pair of equations $u = 0$, $U = 0$, and by $\xi = \eta$, the pair $u = v$, $U = V$.

If the functions U, V are permutable, *i. e.*, if

$$\int U(r, \tau)\, V(\tau, s)\, d\tau = \int V(r, \tau)\, U(\tau, s)\, d\tau,$$

then the operation defined by (2) is commutative, and ξ and η are said to be permutable.

If $u = 0$, ξ is said to be a function of nullity. We see, by referring to the condition for equality, that division by a function not of nullity is equivalent to solving a linear integral equation of the second kind of Fredholm or Volterra type, and that division by a function of nullity is often impossible. With this distinction, if our quantities are permutable among themselves we may apply *all* the laws of algebra, and if the quantities are not permutable, all except the commutative law.* The function of nullity, which includes zero as a special case, takes much the same place in this algebra as the quantity zero in the ordinary algebra. In fact the two algebras are merihedrically isomorphic.† If we do not divide by zero in the ordinary algebra, we do not need to divide by a function of nullity in the symbolic correspondence.

2. If now we introduce into the quantities u and U which define ξ extra parameters or variables $(x_1, x_2, \cdots)$ we have obviously the formula

$$\frac{\partial}{\partial x_i}(\xi\eta) = \xi\frac{\partial \eta}{\partial x_i} + \frac{\partial \xi}{\partial x_i}\eta \tag{3}$$

irrespective of the commutativity or non-commutativity of the product $\xi\eta$. To every differential equation corresponds then an integro-differential equation by means of this symbolism, and a large part of the theory of differential equations may be carried over word for word to these other equations, even when the multiplication is not commutative.

It may be noticed, however, that this symbolism, in particular equation (3), is valid for those integro-differential equations only in which the variables of differentiation $(x_1, \cdots)$ are different from those of integration r, s. These equations were called by their inventor *"of static type,"* on account of

---

* Evans, *L'algebra delle funzioni permutabili e non permutabili*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), pp. 1–28, and ibid. vol. 35, Errata Corrigi.

† Evans: *Sopra l'algebra delle funzioni permutabili*, Atti delle R. Accademia dei Lincei, ser. 5*a*, vol. 8 (1911), p. 710.

their application to the theory of slow motion.* It is the object of this paper to extend Cauchy's theorem for differential equations to integro-differential equations of this type.†

Theorem V, in the paper of Volterra first cited, deals with equations of this kind, and demonstrates the existence of what correspond, in the theory of differential equations, to the general and complete integrals of the equation. Boundary problems for this class of equations have been discussed by Volterra, Lauricella, recently by J. Pérès and others. Integro-differential equations and functional equations of non-static type have also been discussed by Volterra and others, and some corresponding existence theorems have been obtained.

Systems of symbolic differential equations with permutable coefficients

3. Let us limit ourselves to two variables of differentiation x, y besides the variables of integration r, s, and let us consider the system of symbolic equations

$$(4)\qquad \begin{aligned} \frac{\partial \xi_1}{\partial y} &= F_1\left(\frac{\partial \xi_1}{\partial x}, \cdots, \frac{\partial \xi_n}{\partial x}; \xi_1, \cdots, \xi_n; y, x \mid r, s\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \frac{\partial \xi_n}{\partial y} &= F_n\left(\frac{\partial \xi_1}{\partial x}, \cdots, \frac{\partial \xi_n}{\partial x}; \xi_1, \cdots, \xi_n; y, x \mid r, s\right). \end{aligned}$$

The variables r, s are to be restricted to a single region T of the complex plane, such that if r and s are any two points of T the whole of the straight line joining them lies in T. If for $x, y, \xi_1, \cdots, \xi_n, \partial\xi_1/\partial x, \cdots, \partial x_n/\partial x$ we substitute variables $x_1, \cdots, x_{2n+2}$, the functions $F_1, \cdots, F_n$ are to be analytic in these variables in a $(2n+2)$-dimensional neighborhood of the origin, which may be chosen in such a way as to be independent of r, s. The coefficients in the developments of these F_i are continuous functions of r, s in T, permutable among themselves.

The developments in (4) will be convergent and have meaning provided that the ordinary coefficients of the ξ_i, with their first derivatives in x, lie in the given $(2n+2)$-dimensional neighborhood.‡

* V. Volterra: *Sulle equazioni della elettrodinamica*, Rendiconti della R. Accademia dei Lincei, vol. 23 (1909), p. 209.

† In the interim of the writing of this paper and its publication, an article has appeared by M. Paul Lévy, in which the notion of characteristic is applied to another kind of functional equation. See: *Sur l'intégration des équations aux dérivées fonctionelles partielles*, Rendiconti del Circolo Matematico di Palermo, vol. 37 (1914), pp. 113–169.

‡ See: *A theorem of convergence*, Rendiconti del Circolo Matematico di Palermo, vol. 34 (1912), p. 10. To insure this convergence when the limits of integration are constant, parameters $\lambda_1, \cdots, \lambda_n$ must be introduced into the j-coefficients of $\xi_1, \cdots, \xi_n$. In Theorem 1, however, on account of the initial conditions these parameters are provided for by the variable y.

THEOREM 1. *There is one and only one system of solutions* $\xi_1, \cdots, \xi_n$ *of the system of equations* (4) *analytic in* x *and* y *in the neighborhood of the origin, and such that for* $y = 0$ *the conditions* $\xi_1 = 0, \cdots, \xi_n = 0$ *are satisfied. These solutions are themselves permutable with the coefficients of* $F_1, \cdots, F_n$ *and with each other.*

From the fact that the region of analyticity of the F_i is independent of r, s we are able to get inequalities for the values of the coefficients in the development of the F_i. We have then a complete identity of operations with the corresponding theory for differential equations; whence is established, step by step, the permutability of every coefficient in the development of the ξ_i with those of the F_i, and also the fact that they are permutable with each other, because they are, symbolically, integral rational expressions in the coefficients of the F_i. Likewise is established the uniqueness of the formal solutions so developed. The convergence of the developments follow immediately from the convergence of the corresponding power series for the solution of the analogous differential equations if we replace the length of the rectangle of convergence ρ, in the y direction, in the theory of differential equations, by the distanec ρ / c, where c is some quantity greater than unity, and greater also than the interval of integration, ab, or rs, as the case may be.

In fact, if

$$| \xi | = | u + jU | = | u | + | U | < A,$$

$$| \eta | = | v + jV | = | v | + | V | < B,$$

it follows that

$$| \xi\eta | < ABc,$$

and these are the inequalities that it is necessary to use in the demonstration of the convergence of the j-coefficients and ordinary coefficients in the developments of the ξ_i.

4. Let us say that a function $u(x \mid r, s)$

$$u(x \mid r, s) = \sum_0^\infty {}_i A_i(r, s)(x - x_0)^i$$

s *permutably analytic*, or permutably analytic in x, if it is analytic in x and all its coefficients are mutually permutable functions of r, s. We notice that if such a function is permutably analytic about the point x_0, it is permutably analytic when developed about any other point x_1 in the region of analytic extension; for the new coefficients are linear expressions in terms of the old, the coefficients in these expressions being monomials in $x_1 - x_0$.

A necessary and sufficient condition that $u(x \mid r, s)$ be permutably analytic in a given region is that it be analytic in that region and satisfy the condition

$$\int u(x_1 \mid r, \tau)\, u(x_2 \mid \tau, s)\, d\tau = \int u(x_2 \mid r, \tau)\, u(x_1 \mid \tau, s)\, d\tau,$$

where the limits of integration are constant, or s, r, as we deal with one kind of permutability or the other, and where x_1 and x_2 are any two values within the region of analyticity.*

The solutions $\xi_1, \cdots, \xi_n$ *in Theorem* 1 *are permutably analytic.*

We say that two functions are *analytically permutable*, or analytically permutable with each other, when every coefficient in the development of one is permutable with each of the coefficients of the other. As before, this definition is independent of the point x_0, or (x_0, y_0), about which the functions are developed.

Two functions are permutable if they are analytically permutable.

A necessary and sufficient condition that $u(x \mid r, s)$ and $v(x \mid r, s)$ be analytically permutable in x is that they be analytic in x in two regions σ_1 and σ_2 respectively, and that

$$\int u(x_1 \mid r, \tau)\, v(x_2 \mid \tau, s)\, dt = \int v(x_2 \mid r, \tau)\, u(x_1 \mid \tau, s)\, d\tau,$$

where x_1 is any value in σ_1 and x_2 any value in σ_2.

5. We may generalize Theorem 1 to the case where $\xi_1, \cdots, \xi_n$ are, for $y = 0$, arbitrary functions of x, r, s provided that these functions are permutably analytic in x, and analytically permutable with each other and with the coefficients in the developments of the F_i.† The F_i are assumed to be developable throughout a $(2n + 2)$-dimensional neighborhood that includes the ordinary coefficients of the given values of the ξ_i. *There is a unique system of solutions, and these solutions are permutably analytic, and analytically permutable with each other and with the coefficients of the* F_i.

Systems of symbolic differential equations whose coefficients are not necessarily permutable

6. As has been remarked by M. Pérès in regard to certain integral equations,‡ the method of procedure may be extended to take care of the case where the coefficients in the analytic developments are not necessarily permutable among themselves. We have, as before, a unique formal determination of the coefficients, and the convergence of the resulting series is established§ by

* Compare the concept of "continuous group of permutable functions," V. Volterra: *Sopra equazioni di tipo integrale*, Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1913, vol. 1, p. 405.

† When the limits of integration are constants, the given values of the j-coefficients of $\xi_1, \cdots, \xi_n$ are supposed to contain parameters $\lambda_1, \cdots, \lambda_n$ respectively, unless the F_i are entire functions of the $2n$ arguments consisting of the ξ_i and their derivatives.

‡ J. Pérès: *Sulle equazioni integrali*, Rendiconti della R. Accademia dei Lincei, ser. 5*a*, vol. 22 (1913), pp. 66–70.

§ The system (4) may be replaced by one linear in the derivatives in regard to x by the introduction of n more equations and n more unknowns. The dominant series is most simply stated in terms of the linear system.

means of exactly the same dominant series that is used in Theorem 1. In fact, the less-or-equal relation between the terms of series and dominant series (all of whose coefficients are positive) holds even to the partial terms whose sum is the coefficient of $x^m y^n$.

Let F be analytic in its $2n+2$ arguments in a $(2n+2)$-dimensional neighborhood which includes the points determined by the ordinary coefficients of the functions $\xi_1(x \mid r, s), \cdots, \xi_n^0(x \mid r, s)$, functions which are analytic in x, in the neighborhood of $x = 0$, with coefficients that in T are continuous functions of r, s. We have then the following theorem:

THEOREM 2. *There is one and only one system of solutions* $\xi_1, \cdots, \xi_n$ *of* (4) *under the given conditions,* which are analytic at the origin in* x *and* y, *and which, for* $y = 0$, *take the given values* $\xi_1^0, \cdots, \xi_n^0$.

By means of the process of analytic extension we may extend our theorem to the case where the given functions are analytic along a certain given segment of the x-axis.

7. Two of Volterra's integro-differential equations, viz.,

$$\nabla^2 u(x, y \mid r, s) + \int_y^x \left\{ A_1(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial x^2} + A_2(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial y^2} \right\} d\tau = 0 \tag{5}$$

and

$$\nabla^2 u(x, y \mid r, s) + \lambda \int_a^b \left\{ A_1(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial x^2} + A_2(r, \tau) \frac{\partial^2 u(x, y \mid \tau, s)}{\partial y^2} \right\} d\tau = 0 \tag{6}$$

have the common symbolic form

$$\alpha_1 \frac{\partial^2 \xi}{\partial x^2} + \alpha_2 \frac{\partial^2 \xi}{\partial y^2} = 0, \tag{7}$$

in which are to be considered only the solutions of nullity.

For small values of λ, the equation (7) may be written in the form

$$\frac{\partial^2 \xi}{\partial y^2} = \alpha \frac{\partial^2 \xi}{\partial x^2}, \tag{8}$$

in which

$$\alpha = -\frac{1}{\alpha_2} \alpha_1.$$

In close relation to (8) stands the system of equations

* As in § 5, parameters $\lambda_1, \cdots, \lambda_n$ must be introduced when the limits of integration are constant.

$$(9) \qquad \frac{\partial \xi_2}{\partial y} = \alpha \frac{\partial \xi_1}{\partial x}, \qquad \frac{\partial \xi}{\partial y} = \xi_2, \qquad \frac{\partial \xi_1}{\partial y} = \frac{\partial \xi_2}{\partial x}.$$

Under the boundary conditions

$$(9') \quad (\xi)_{y=0} = \xi^0(x, r, s), \quad (\xi_2)_{y=0} = \xi_2^0(x, r, s), \quad (\xi_1)_{y=0} = \frac{\partial}{\partial x}\xi^0(x, r, s),$$

the system (9) has one and only one analytic solution.*

But the system (9) under the conditions (9′) is equivalent to the system (8) under the conditions

$$(8') \qquad (\xi)_{y=0} = \xi^0(x, r, s), \qquad (\partial \xi / \partial y)_{y=0} = \xi_2^0(x, r, s),$$

and (8), then, under these conditions, has one and only one analytic solution. If ξ^0 and ξ_2^0 are functions of nullity, the solution is a function of nullity, and (8) is equivalent to (5) or (6).

Equation (5) needs no further discussion. In equation (6), however, we made the assumption that λ was small. If λ is not small (6) is equivalent to (8) under the condition that λ is not a special parameter value for $A_2(r, s)$. *Equation* (6) *has therefore a unique analytic solution under the given conditions, provided that* λ *is not a special parameter value for* $A_2(r, s)$.

The correspondence between symbolic differential and integro-differential equations

8. Let us consider the equation

$$(10) \qquad \Phi\left(U, \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial^2 U}{\partial x^2}, \cdots; y, x \mid r, s\right) = 0,$$

of which the left hand member is an integral rational expression in U and its derivatives up to the nth order, with coefficients that are analytic in x and y throughout a neighborhood σ, independent of r, s, of a given analytic curve σ. The coefficients in the analytic developments of these coefficients are continuous functions of r, s in T. Multiplication of U by itself or by any of its derivatives, or of any of these by a function of r, s is to be interpreted as combination according to the formula (1) [§ 1]. Multiplication of U or any of its derivatives by a function not involving r, s may be interpreted in the ordinary way; or, on the other hand, that function may be regarded as implicitly a function of r, s, according as we care to consider one equation or another. In this way we generate an integro-differential equation which includes that discussed by Volterra in the Theorem V already referred to.

Equation (10) may be rewritten as a symbolic differential equation. In

* It is not necessary to introduce parameters λ_i into these boundary conditions (see note to § 5).

fact if we replace in it U and its derivatives by $jU = \xi$ and its derivatives, and replace every function involving r, s, explicitly, or by definition, by j times that function, leaving the functions not involving r, s as they are, and if we then collect terms, we shall have a symbolic differential equation whose solutions of nullity are the solutions of (10), and vice-versa. Let us write this equation as

$$\Psi\left(\xi, \frac{\partial \xi}{dx}, \frac{\partial \xi}{dy}, \frac{\partial^2 \xi}{\partial x^2}, \cdots; y, x \mid r, s\right) = 0. \tag{11}$$

9. If the curve σ is the x-axis, and (11) can be solved for $\partial^n \xi / \partial y^n$, the equation (11) may be replaced by a system of equations of the type of (4), and Theorem 1 or Theorem 2 may be applied. We thus are able to restate for integro-differential equations of the form (10) the ordinary Cauchy existence theorems for differential equations.

If the coefficients in the analytic development of (10) are permutable among themselves, the solution of (10) is permutably analytic, and analytically permutable with the coefficients, provided that the same statement is true of the initial conditions.

Characteristics of integro-differential equations*

10. If the curve σ is not the x-axis but is some analytic curve $y = \varphi(x)$, we may still determine a solution of the equation (10), or, what is the same thing, a solution of nullity of equation (11), by assigning proper values for the solution and its derivatives up to and including those of order $n - 1$ along the curve. For, by a transformation of variable, this case may be reduced to that already discussed. For certain families of curves, however, as with differential equations, this transformation is impossible; the solution is not uniquely determined by means of the given conditions. These families of curves, to which correspond the characteristics of differential equations, are the subject of this last section.

Without any essential loss of generality, for our treatment, we can assume that equations (10) and (11) are of the second order. We shall suppose them, for the present, to be linear in the derivatives of highest order. Equation (11) may then be rewritten as

$$\alpha_{11} \frac{\partial^2 \xi}{\partial x^2} + \alpha_{12} \frac{\partial^2 \xi}{\partial x \partial y} + \alpha_{22} \frac{\partial^2 \xi}{\partial y^2} + \lambda = 0. \tag{12}$$

In this equation $\alpha_{11}, \alpha_{12}, \alpha_{22}$ and λ are functions of $\xi, \partial\xi / \partial x, \partial\xi / \partial y, x, y, r,$

* This treatment may be compared with that in Hadamard's *Leçons sur la propogation des ondes*, chap. 7, Paris, 1913. The equations (12), (13), (14), (15), (17), (18′), below, correspond to the equations (1), (7), (8), (8′) and (12), (11), (13) respectively in the chapter cited.

s of the type specified in § 8. We assume that on the curve $y = \varphi(x)$ we are given ξ and $\partial\xi / \partial y$, or what amounts to the same thing, ξ, $\partial\xi / \partial x$ and $\partial\xi / \partial y$ as analytic functions of x in the region σ, with coefficients continuous functions of r, s in T.*

Precisely as in the theory of differential equations, we have from (12), by expressing $\partial^2 \xi / \partial x^2$ and $\partial^2 \xi / \partial x \partial y$ for points on $y = \varphi(x)$ in terms of $\partial\xi / \partial y$ and $\partial^2 \xi / \partial y^2$, as the equation for the determination of $\partial^2 \xi / \partial y^2 = \xi_{22}$

$$\Gamma \xi_{22} + \Pi = 0. \tag{13}$$

In this equation

$$\Gamma = \alpha_{11}\left(\frac{dy}{dx}\right)^2 - \alpha_{12}\left(\frac{dy}{dx}\right) + \alpha_{22},$$
$$\Pi = \alpha_{11}\left(\frac{d\xi_1}{dx} - \frac{dy}{dx}\frac{d\xi_2}{dx}\right) + \alpha_{12}\frac{d\xi_2}{dx} + \lambda, \tag{14}$$

where d / dx refers to differentiation along the curve, and ξ_1 and ξ_2 denote $\partial\xi / \partial x$ and $\partial\xi / \partial y$ respectively. In these formulæ it is important to preserve the order of all quantities that involve the j, so as not to necessitate the introduction of the hypothesis of permutability.

The equation (13) *enables us to determine* ξ_{22}, *unless* Γ *is a function of nullity at some point of the curve* $y = \varphi(x)$; *i. e., unless*

$$\text{Ord } \Gamma = 0. \tag{15}$$

The curves defined by the differential equation (15) may be called the *ordinary characteristics* of the integro-differential equation (10). On account of the way (11) is formed from (10), equation (15), which involves no j-coefficients, must be independent of the solution $\xi = jU$. *The ordinary characteristics are independent of the solution of the equation* (10).

11. If we make an analytic transformation of the independent variables x, y that reduces the curve $y = \varphi(x)$ to the x-axis, it is immediately verifiable that a sufficient condition that the transformation of the equation (11) be solvable for $\partial^n \xi / \partial y^n$ is that the curve $y = \varphi(x)$ be nowhere tangent to an ordinary characteristic. That it is not also a necessary condition depends on the fact that it is often possible to divide by a function of nullity.

THEOREM 3. *If* $y = \varphi(x)$ *is not tangent to any ordinary characteristic, the given values of* U *and* $\partial U / \partial y$ *uniquely determine an analytic solution of* (10).

12. It may happen that all the curves in the plane are ordinary characteristics of the equation (10). In other words Ord Γ may vanish identically. A

* The j-coefficients of these functions of r, s and of the functions and values arbitrarily assigned are supposed to contain parameters λ, if we are dealing with integration with constant limits.

necessary and sufficient condition that Ord Γ be identically zero is that we have identically

$$\text{Ord } \alpha_{11} = \text{Ord } \alpha_{12} = \text{Ord } \alpha_{22} = 0. \tag{16}$$

We may still, however, be able to determine solutions of the equation by giving arbitrary values of U and $\partial U / \partial y$ along $y = \varphi(x)$.

As an illustration of this fact, let us restrict ourselves to an integro-differential equation in which the limits of integration are s, r, and let

$$\alpha_{ij} = jA_{ij}(x, y \mid r, s) + jB_{ij}(x, y \mid r, s)\beta_{ij}, \quad \lambda = jC(x, y \mid r, s)\gamma,$$

where β_{ij}, γ are functions of ξ, $\partial\xi / \partial x$, $\partial\xi / \partial y$, x, y, r, s of the kind already described, and A_{ij}, B_{ij}, C and their first derivatives in regard to r are continuous functions of r, s and analytic functions of x, y, which for $s = r$ become merely functions a_{ij}, b_{ij}, c, of x, y alone. We assume, moreover, that the β_{ij} are functions of nullity.

To investigate this equation, differentiate it with regard to r. It will then, as is directly verifiable, take the form

$$\left\{ a_{11} + j\frac{\partial A_{11}}{\partial r} + \left(b_{11} + j\frac{\partial B_{11}}{\partial r} \right)\beta_{11} \right\} \frac{\partial^2 \xi}{\partial x^2}$$
$$+ \left\{ a_{12} + j\frac{\partial A_{12}}{\partial r} + \left(b_{12} + j\frac{\partial B_{12}}{\partial r} \right)\beta_{12} \right\} \frac{\partial^2 \xi}{\partial x \partial y}$$
$$+ \left\{ a_{22} + j\frac{\partial A_{22}}{\partial r} + \left(b_{22} + j\frac{\partial B_{22}}{\partial r} \right)\beta_{22} \right\} \frac{\partial^2 \xi}{\partial y^2} + \lambda' = 0.$$

The solutions of nullity of this last equation are solutions of nullity of the original one, and vice-versa. But this last equation has definite ordinary characteristics unless a_{11}, a_{12}, a_{22} are all zero.

13. Let us now suppose that we have an integro-differential equation (12) for which the ordinary characteristics are defined, and that $y = \varphi(x)$ is one of them. Let us consider how much is arbitrary in the solution.

The equation (13) shows that it is not generally possible to assign values of ξ and ξ_2 arbitrarily along an ordinary characteristic. For if, for instance, we restrict ourselves to variable limits of integration r, s and take account of the fact that since $y = \varphi(x)$ is an ordinary characteristic Γ is a function of nullity, it follows that in general $\Gamma\xi_{22}$ vanishes to a higher order than Π along $r = s$, and equation (13) is not satisfied. By properly choosing the values of ξ and ξ_2, however, we may still have solutions.

By successive differentiation of (12) and elimination of all partial derivatives except those with regard to y, by means of the relation $y = \varphi(x)$, we have the equation

$$\Gamma\xi_{222} + \Pi_1 = 0, \qquad \Gamma\xi_{2222} + \Pi_2 = 0, \qquad \cdots, \tag{17}$$

in which

$$\Pi_i = \left(\alpha_{12} - 2\alpha_{11}\frac{dy}{dx}\right)\frac{d}{dx}\left(\frac{\partial^{i+1}\xi}{\partial y^{i+1}}\right) + \mu_i, \tag{18}$$

where μ_i does not involve differentiation along $y = \varphi(x)$ of $\partial^{i+1}\xi / \partial y^{i+1}$. Hence, if all the derivatives of ξ of order up to i are known, Π_i becomes an integro-differential expression of the first order in $\partial^{i+1}\xi / \partial y^{i+1}$. We may also write

$$\Pi = \alpha_{11}\left(\frac{d^2\xi}{dx^2} - 2\frac{dy}{dx}\frac{d\xi_2}{dx} - \frac{d^2y}{dx^2}\xi_2\right) + \alpha_{12}\frac{d\xi_2}{dx} + \lambda. \tag{18'}$$

Let us assume that we are able to choose our arbitrary values so that

$$j\text{-coefficient } \Gamma = 0. \tag{19}$$

From (13) it follows that necessarily

$$\Pi = 0, \tag{20}$$

and from (17),

$$\Pi_1 = 0, \quad \Pi_2 = 0, \quad \cdots. \tag{21}$$

We must then, if we choose values to satisfy (19), so choose them that they also satisfy (20) and (21).

We may regard (19) and (20) as simultaneous equations that hold along $y = \varphi(x)$ for ξ and ξ_2. In fact, if in the expression for j-coefficient Γ we substitute for the ξ_1 that appears in α_{11}, α_{12}, α_{22}, by means of the formula

$$\xi_1 = \frac{d\xi}{dx} - \frac{dy}{dx}\xi_2,$$

we get for (19) an integro-differential equation of order zero in ξ_2 and of the first order in ξ. Considered as an equation to determine ξ_2, it is an integral equation, and may generally be solved for ξ_2. On substituting this value in (20), we have an integro-differential equation of the second order to determine ξ, simpler than those discussed here, since it involves differentiation with regard to but one variable. In general, then, *if* (19) *holds we may assign at one point of the characteristic arbitrary values of* ξ *and* $d\xi / dx$, *whence will be determined the values of* ξ *and* ξ_2 *all along the curves.* The equations $\Pi_1 = 0$, $\Pi_2 = 0$, $\cdots$ now become integro-differential equations of the first order to determine ξ_{22}, ξ_{222}, $\cdots$ respectively. Therefore *we may assign arbitrarily the values of* ξ_{22}, ξ_{222}, $\cdots$ *at one point of the characteristic, whence their values will be determined all along the curve.*

14. These results may be generalized in various ways. The extension is immediate to the case where instead of x, y we have n variables x_1, x_2, $\cdots$, x_n. The characteristics become spaces of $n - 1$ dimensions,

$$x_n = f_i(x_1, x_2, \cdots, x_{n-1}),$$

and provided that the given $(n-1)$-space

$$x_n = \varphi(x_1, x_2, \cdots, x_{n-1})$$

contains no $(n-1)$-dimensional element of a characteristic, arbitrarily assigned values of ξ and $\partial\xi/\partial x_n = \xi_n$ determine a solution of the integro-differential equation throughout the n-space. In a characteristic space the values of ξ and $\partial\xi/\partial x_n$ may no longer be chosen arbitrarily. If they are taken so that (19) is satisfied, then along an $(n-2)$-space

$$x_{n-1} = \Psi(x_1, x_2, \cdots, x_{n-2})$$

values of ξ, $\partial\xi/\partial x_{n-1}$, ξ_{nn}, $\cdots$ may be chosen arbitrarily, their values being then determined by the integro-differential equation throughout the given $(n-1)$-space. Here by $\partial\xi/\partial x_{n-1}$ we mean the total derivative in the $(n-1)$-space. In fact, the equation (19) may be rewritten as an integral equation in ξ_n, as before it was rewritten for ξ_2, and the result of substituting in (20) is an integro-differential equation of the second order in ξ. This equation and the equations (21) are integro-differential equations of the kind treated in this article, referred to $n-1$ variables of differentiation. Thus a formal development for a solution is obtained.

In the extension of the results for equations of the second order to those of higher order, or from equations linear in the derivatives of highest order to those that are not, there is no difference in the case that we are now treating from the similar extensions in the case of differential equations.

BOSTON, *July* 26, 1913.